ELSEVIER

Learning and Instruction 14 (2004) 519–534

Learning and Instruction

www.elsevier.com/locate/learninstruc

Number concept and conceptual change: towards a systemic model of the processes of change

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Abstract

The research on conceptual change has so far mainly dealt with cognitive outcomes, but especially during the last few years there has been a growing interest in and discussion about the processes of conceptual change. The purpose of the article is to contribute to this discussion and to present a theoretical model of the dynamics among the cognitive and motivational factors in conceptual change. Several researchers in science education have proposed cognitive conflict as instructional strategy for teaching difficult scientific concepts. However, we aim to explain why it does not always support the conceptual change. In our model, the two crucial aspects to the process of conceptual change are: the sensitivity to the novel aspects in the situation, and the ability to regulate the tolerance of ambiguity resulting from the experience where the prior knowledge is not adequate.

Keywords: Conceptual change; Motivation; Metacognition; Cognitive distance; Tolerance of ambiguity; Certainty; Number concept

1. Introduction

Conceptual change is used to characterize situations where learners' prior knowledge is incompatible with the new conceptualization, and where learners are often disposed to make systematic errors or build misconceptions. During the last years, research on conceptual change has produced a rich variety of models of the development of students' conceptual understanding and conceptual change (Carey,

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0959-4752/$ - see front matter
doi:10.1016/j.learninstruc.2004.06.016

1985; Carey & Spelke, 1994; Chi, Slotta, & de Leeuw, 1994; Duit, 1999; Hatano & Inagaki, 1998; Karmiloff-Smith, 1995; Vosniadou, 1994, 1999; Vosniadou & Brewer, 1994). Researchers (e.g. Schnotz, Vosniadou, & Carretero, 1999; Vosniadou, 1994) make a distinction between different qualities of learning processes targeted at conceptual change: continuous growth and discontinuous change. The easier level of learning is called *enrichment*, suggesting continuous growth, improving the existing knowledge structure, etc. Discontinuity of learning occurs in a situation where prior knowledge is incompatible with the new information and needs *reorganization*; where significant restructuring—not merely enrichment—of existing knowledge structures is needed. This kind of knowledge acquisition is typical in specific domains of science (Posner, Strike, Hewson, & Gertzog, 1982). However, the results of several empirical studies have indicated that students' prior knowledge is often resistant to teaching efforts. Another important problem is that students are usually not aware of the quality of their previously acquired knowledge and its contradictions with scientific knowledge. Thus, because they do not see or understand any reasons for change, they tend to ignore or to enrich their prior representations rather than revise them (Duit, Roth, Komorek, & Wilbers, 2001; Guzzetti, Snyder, Glass, & Gamas, 1993; Vosniadou, 1999). Recent empirical research has suggested that conceptual change is a very complex process that proceeds through the gradual replacement of prior beliefs and presumptions (e.g. Vosniadou, 2003; Vosniadou, Ioannides, Dimitrakopoulou, & Papademetriou, 2001).

So far empirical research on conceptual change has mainly dealt with cognitive factors, but especially during the last few years there has been a growing interest in and discussion about the dynamics of the processes themselves. Several researchers have agreed that these processes cannot be explained in mere cognitive terms, but that also motivational factors (e.g. Linnenbrink & Pintrich, 2003; Pintrich, 1999; Pintrich, Marx, & Boyle, 1993), metacognitive factors (Flavell, 1979; Hacker, 1998; Schoenfeld, 1987), and meta-conceptual awareness (e.g. Mason, 2001; Mason & Boscolo, 2000; Vosniadou, 1999) should be considered.

The aim of this article is to contribute to the above-mentioned discussion by presenting a theoretical model of the processes of conceptual change. The model is based on studies conducted in our lab on learning disabilities (Lehtinen, 1984), and on the radical conceptual change involved in the extension of the number concept (see Merenluoto & Lehtinen, 2002). This type of change shares many of the features of conceptual change in other fields, like physics, but there are also characteristics typical of mathematics. In the pages that follow, we briefly summarize these findings, and then present a theoretical model that attempts to captive the dynamics among cognitive, metacognitive, and motivational processes in conceptual change. The model also aims to explain why cognitive conflict does not always support conceptual change. Finally, we present results from our ongoing research on conceptual change in mathematics that support the model.

2. Conceptual change in the extension of the number concept

One of the very first extensions of the number concept in formal mathematics learning is the change from operations carried out with natural numbers (1, 2, 3, ...) to operations where rational numbers are used. The essence of natural numbers is their discrete nature, which means that for every number a successor is defined, and no two numbers have the same successor. But for rational numbers a successor is not defined because the rational numbers are defined as a relation of two integers (where the denominator is not a zero) in which infinite successive division is possible. Thus, the underlying but fundamental difference between these number domains is the discrete and dense[1] (continuous) nature of numbers. Although the advanced properties of rational numbers are not explicitly taught at the lower levels of mathematics education, these properties are embedded in the representations, the rules of operations and of order for these numbers, and they are essentially different compared to the respective rules of natural numbers. Thus, every extension of the number concept requires new rules to be learned for operations as well as the use of a new kind of logic, often leading to many different, but systematic problems and misconceptions in mathematics learning (c.f. The multiplier effect, see Verschaffel, De Corte, & Van Coillie, 1988).

Small natural numbers and the concept of a successor are among those special concepts which have a high unconditional certainty attached to them. This kind of certainty seems to be derived from at least three different sources. First, in several empirical studies it has been found that even infants have an intuitive conception of small cardinalities as discrete objects (e.g. Starkey, 1992; Starkey, Spelke, & Gelman, 1990), suggesting that humans have an innate cognitive mechanism related to numeral reasoning principles (Gallistel & Gelman, 1992). The second source of this the certainty comes from everyday experiences and linguistic operations (Wittgenstein, 1969) in counting objects. Later, in formal mathematics instruction, these prior concepts of discreteness are strengthened, and it is important that they are strengthened in order to teach and learn the notion of natural numbers. Thus, the conception of the discrete nature of numbers is based on innate cognitive mechanisms, powerful experiences of everyday counting, and on formal mathematics instruction. Hence, we claim that the discrete nature of numbers is one of the basic ontological presumptions and epistemological beliefs of the naive framework theory of numbers. Because of the early intuitive feeling for the discreteness of small cardinalities, and the abundant everyday experiences of the next object in the counting process, we claim that the change from the use discrete natural numbers to the use of rational numbers requires a radical conceptual change for the learner.

In fact, this kind of conceptual change is especially demanding because, as it is typical in mathematical knowledge, objects at the lower level of the hierarchy are not forsaken in the construction process, but integrated into larger conceptualiza-

[1] With the term “dense” we mean rational numbers as a dense subset of real numbers, where between any two there are infinity of numbers.

tions to create a coherent whole. In the process of conceptual change the natural numbers are not replaced by the rational numbers, but an alternate more abstract structure for numbers needs to be constructed. The fact, the notion of a successor is still always valid whenever operating with natural numbers and integers can be confusing for the learner. The process of extending of the number concept to rational numbers requires meta-conceptual awareness of the differences in the two kinds of numbers, conscious thinking in constructing a parallel mental model for numbers, and metacognitive control in the selection to appropriate rules of operation depending on the task at hand. In the later development of conceptual change in the number concept, the dense set of rational numbers creates a new frame of reference for numbers, from which mapping to natural numbers is formed and they treated as a subset of rational numbers. For example, when the number 2 is treated as a natural number, it represents two objects, but when treated as a rational number it gives a relation of 2 to1. In fact this kind of change in the frame of reference of numbers is a continuous process of vertical hierarchical abstractions (e.g. Landau, 1951/1960; Dreyfus, 1991), as rational numbers will later be treated as subsets of real numbers, with the process of extension continuing with complex numbers, etc. For mathematicians, the hierarchical construction of numbers is logical and coherent because they are already familiar with the structure. For students, however, it looks fragmented and inconsistent because they do not have enough structural knowledge to recognize the logic in the hierarchical system of numbers.

Even at the higher levels of education, students seem to be unaware of their thinking about numbers or of the fundamental conceptual difference between natural and rational numbers (Merenluoto & Lehtinen, 2002). Because of the operational justification of the extension of the number concept, little attention is paid to the underlying general principles of the different number domains in the traditional curriculum. The natural numbers and their discrete nature have a high intuitive acceptance attached to them as being self-evident, self-justifiable or self-explanatory, and this easily results in over-confidence (Fischbein, 1987).

The resilience nature of prior thinking about numbers and the tendency towards over-confidence were demonstrated in the results of several surveys we have done to test students' basic understanding of real numbers and to analyze the quality of the students' conceptual change. Participants in these surveys were students at upper secondary level ($n = 538$) from 24 randomly selected Finnish upper secondary schools (Merenluoto & Lehtinen, 2002), student teachers ($n = 62$) and students majoring in mathematics at university level ($n = 71$). A great majority of the student teachers and more than one-tenth of the students majoring in mathematics at the university level consistently used the rules ("add one…") of discrete numbers when they were asked to explain how many fractions or real numbers there were between two given numbers, and which number was the "next" after or "the closest" to a given number (Fig. 1).

The conflict between the discrete and dense was explicitly seen in the wavering explanations, where the students, on the one hand, wrote "*it is not possible to define the next number*" but, on the other hand, continued believing in its actual existence: "*…but it is the one which is the closest", "…the one which has the most*

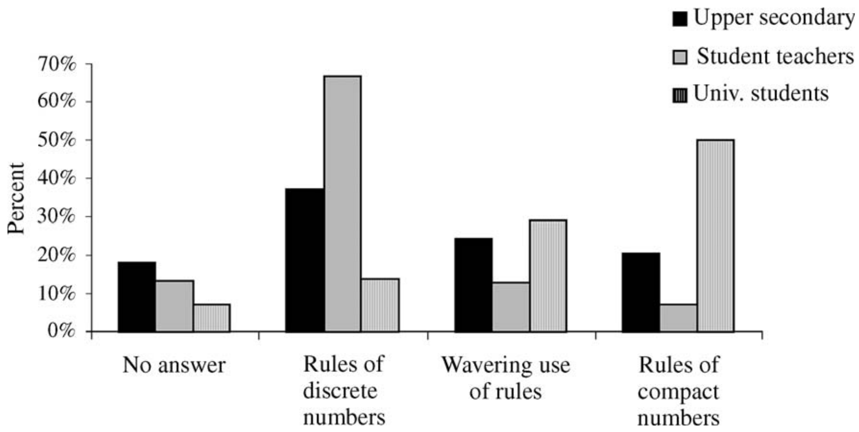

Fig. 1. The percentage distribution of students' use of rules of discrete and compact numbers in their explanations of the density of rational and real numbers on the number line.

decimals" (Merenluoto & Lehtinen, 2002: p. 249). The results also suggested that a moderate operational understanding of these concepts has a tendency to lead to over-confidence or to the illusion of understanding, in which case the cognitive conflict passes unnoticed.

3. A theoretical model of the processes of conceptual change

In this section we present a theoretical model of the dynamics of motivational, metacognitive and cognitive processes in conceptual change. By means of the model we aim to explain why cognitive conflict does not always support conceptual change. Cognitive conflict has been popular in many traditional theories of development and learning (see Dewey, 1929, 1933; Flavell, 1977; Piaget, 1978; Vygotsky, 1934/1994), and several authors in science education have proposed cognitive conflict as an instructional strategy for teaching difficult scientific concepts (e.g. Strike & Posner, 1982). However, research findings have shown the use of cognitive conflicts to facilitate radical conceptual change to be controversial (Limón, 2001; Limón & Carretero, 1999; Trumper, 1997).

The starting point in our model is a learning situation, in which the learner meets tasks and materials that include phenomena calling for new conceptual understanding. Our hypothesis is that the perception of the task (task situation) is influenced by students' cognitive, metacognitive, and motivational sensitivity to the task. Sensitivity in this model refers to the extent to which the student is aware of and interested in the novel cognitive aspects of the phenomenon (see also De Corte, Greer, & Verschaffel, 1996). Cognitive and metacognitive sensitivity refers to the relation between learners' prior knowledge and the cognitive demands of the task, and to the meta-conceptual awareness about their thinking about the subject (e.g. Mason, 2001; Mason & Boscolo, 2000). Motivational sensitivity describes the learners' tendency to look for novel and surprising features during learning activities (e.g. having a mastery goal orientation or achievement goal leading to interest and intentional learning, see Linnenbrink & Pintrich, 2003) (Fig. 2).

One possible route in the process of conceptual change for the learner in our model is a situation where his or her prior knowledge is completely insufficient for a relevant perception of the new phenomenon (later called the "no-relevant-perception" path). If the situation is socially or personally important to the learner, this confusing situation might induce a feeling of low certainty and more or less random attempts to fulfil the social expectations of the situation or to activate avoidance behavior. From an instructional point of view, this means that any attempt to create a cognitive conflict would fail to result in the desired conceptual change. It is obvious that the lack of adequate prior knowledge has different meanings in different domains. In some domains (e.g. history, social sciences), it might be easier to develop ad hoc strategies to overcome this knowledge distance, whereas in other areas (e.g. advanced mathematics) insufficient prior knowledge completely prevents the learner from perceiving the task in any relevant way.

Another possible route is a situation where the learner's sensitivity to the novel features of the phenomenon is high. It means that he/she has sufficient prior knowledge to understand the cognitive demands of the task, which go beyond his or her current conceptual understanding, and is disposed to pay attention to the

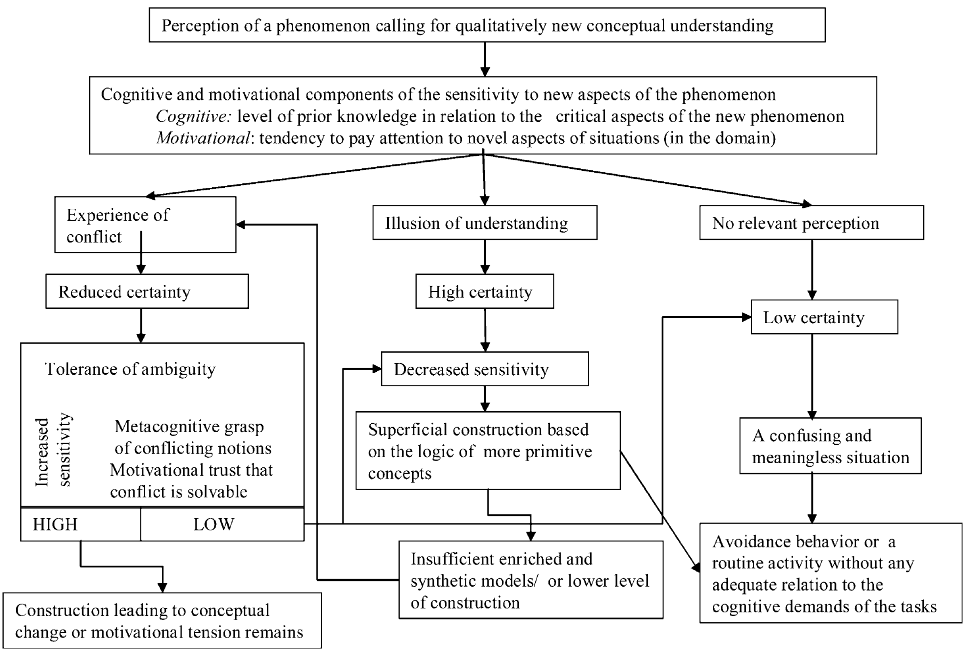

Fig. 2. Theoretical model of the dynamics of cognitive sensitivity, experienced certainty and tolerance of ambiguity in the processes of conceptual change.

unknown features of the situation. In defining both aspects of sensitivity we refer to the findings of research on comparing experts with novices. In their classic study, Chi and Glaser showed that, in new problem situations, novices tend to pay attention to the superficial features of the tasks, whereas experts perceive the tasks in terms of underlying general principles (Chi, Feltovich, & Glaser, 1981; Chi, Glaser, & Farr, 1988). In typical conceptual change problems, the relevant cognitive conflicts lie on the conceptual level and are not necessary observable on the surface level. The latter aspect of sensitivity is similar to the dynamic expertise model of Bereiter and Scardamalia (1993) in which expertise is seen more as an orientation to seek new challenges than as a static state of established skills (see also the adaptive expertise model developed by Hatano, 1982).

The above processes of sensitivity and of the tendency to pay attention to the novel features in the situation are necessary conditions for productively experiencing a cognitive conflict. The experience of a cognitive conflict then results in reduced certainty (e.g. Efklides, Akilina, & Petropoulou, 1999). This means that the learner does not base his or her thinking merely on established knowledge but is prepared to change his or her knowledge beliefs. In many advanced scientific concepts, the construction of higher-level understanding is not based on immediate insight, but instead the long-lasting construction of a new knowledge system is needed. The learner needs to tolerate a cognitive ambiguity when dealing with the new knowledge system (e.g. Sorrentino, Bobocel, Gitta, & Olson, 1988; Stark, Mandl, Gruber, & Renkl, 2002). The concept of the tolerance of ambiguity was first used by Frenkel-Brunswik (1948) and later elaborated by Budner (1962). Originally the tolerance of ambiguity was understood as a personality variable, but here we consider it to be a more specific variable referring also to the domain of the needed change. It can further be explained as a dynamic process of metacognitive and motivational variables. Thus, coping with a new complex conceptual system is possible only if the learner has sufficient metacognitive skills to grasp conflicting notions. This is, however, not enough, but he or she also needs to be motivated to deal with the ambiguity, and to trust that the experienced conflicts are solvable. If the learner's tolerance of ambiguity is high, he or she can remain sensitive to the novel cognitive demands, and even increase the sensitivity during the learning process. Our hypothesis is that this kind of learning approach (the experience-of-conflict path) is needed in tasks that demand conceptual change. If the tolerance of ambiguity is low, this might lead to decreased sensitivity or loss of trust, which results in low certainty and avoidance behavior (cf. the expectancy-value theories of achievement motivation, McClelland, 1985).

Motivation seems to be related to the process of conceptual change in a very complex way. There is no doubt that interest and self-efficacy (e.g. Linnenbrink & Pintrich, 2003; Schoenfeld, 1987) are the fundamental aspects of high sensitivity and high tolerance of ambiguity. However, high interest and self-efficacy might also increase the learner's tendency to take the path of illusion-of-understanding. This means that the learners' prior knowledge is developed enough for recognizing familiar elements in the new phenomenon, but not enough to pay attention to the novel aspect going beyond his/her current conceptions. This arouses an illusion of understanding which results in (unfounded) high certainty. The high certainty decreases the learner's sensitivity to possible conflicting features of the situation and leads to construction of knowledge that is based on logic and concepts that are too primitive in relation to the cognitive demands of the task. Enrichment of naive models or construction of the so-called synthetic models is based on this kind of learning path. If the learner is motivated enough, it is obvious that in the course of time these enriched or synthetic models will bring about an experience of conflict and open possibilities for more radical conceptual change later on.

4. Paths of conceptual change in the number concept

In our theoretical model, one of the key indicators referring to different paths of conceptual change is the experience of certainty in relation to a phenomena calling for new conceptual understanding. In our before mentioned study, 538 students from randomly selected 24 Finnish upper secondary schools participated into a number concept test (see Merenluoto & Lehtinen, 2002). The test consisted of identification, classification and construction problems in the domain of rational and real numbers. There, the real number concept of the students was tested on three main dimensions: formal hierarchy and facts of real numbers, the density (no gaps nature) of the number line, and concepts of the limit and continuity of a function. The students were also asked to estimate their certainty about their answers. We performed a cluster analysis on students' scores and their respective certainty estimations resulting to four different profiles referring to the paths in our model, with the exception that we found two different versions to the path of illusion-of-understanding: one where the central tendency was over-confidence and another with dependence to familiarity. In addition, as seen in Table 1, a student's *tendency to avoidance* was calculated as a percentage of the tasks left unanswered in the test; a student's *tendency to over-confidence* as percentage points by subtracting the task scores from the respective certainty scores; and a student's *group position* by subtracting the respective group mean of the grades in mathematics from the student's individual grades. Thus, a group position close to zero means that a student's achievement level was about average in his or her respective study group.

4.1. Examples of the main paths in the model

The results of profile 1 referred to the *no-relevant-perception* path of the model. These students identified numbers only by their superficial features as whole numbers, fractions and decimal numbers, thus indicating that their prior knowledge was insufficient for the tasks. They consistently used rules of discrete numbers in their answers. Their low achievement level, clearly below the average in their respective study groups, suggests that they had experienced failures in mathematics. Their estimated certainty level was low, suggesting that they were more or less confused. Thus, their motivational tendency to deal with the difficulties of the tasks was mainly avoidance.

Table 1
The means of variables behind the profiles

Profile	$n = 538$ (%)	Achievement level in math	Group position of students[a]	Tendency toward avoidance (%)[b]	Task scores % of maximum[c]	Certainty scores % of maximum[d]	Tendency toward over-confidence[e]
Profile 1	24	Low	−0.49	45	22.6	18.7	−3.9
Profile 2	30	Average	−0.06	33	31.7	32.5	0.8
Profile 3	36	Average	0.13	16	47.1	62.5	15.4
Profile 4	10	High	1.08	0	68.5	66.8	−1.7
All	100		0.01	28	38.6	42.5	3.9

Significant ($p < 0.001$) difference between profiles:
[a] $F(3, 525) = 16.20$, ns. between profiles 2 and 3.
[b] $F(3, 534) = 48.57$, ns. between profiles 3 and 4.
[c] $F(3, 534) = 125.97$.
[d] $F(3, 534) = 120.29$, ns. between profiles 3 and 4.
[e] $F(3, 534) = 44.73$, ns. between profiles 2 and 4.

More than two-thirds of the students represented the *illusion-of-understanding* path of the model, of which we found two different versions. In the first version (profile 2, see Table 1), the students showed context-related certainty estimations, characterised by a slight over-confidence for tasks, which seemed familiar (they had seen them in the textbooks), and low certainty in more difficult tasks (needing spontaneous justification) reflecting a dependence on social evidence. These students mainly used the rules of discrete numbers in their answers. Their cognitive sensitivity to the critical aspects of compact numbers was rather low. The second version of the path (profile 3, see Table 1) was characterised by systematic over-estimation of certainty. The level of students' explanations was significantly better than that of the students in the profile 2. This means that they gave operational level answers to the tasks, such as explaining the infinity of numbers by referring to the possibility of adding decimals, or explaining the continuity of the function with the possibility to draw it "without lifting the pen". In their answers there were no references to the structural differences between the number domains, suggesting an enrichment type of change, which combined with their high certainty estimations, suggested an illusion of understanding.

The students in profile 4 (see Table 1) represented the *experience-of-conflict* path in the model. These students were high-achieving students in mathematics, above average in their respective study groups. Their explanations indicate a metacognitive grasp of conflicting notions; in fact, they consistently identified the abstraction of limit in explaining the density of numbers, like "it is not possible to define the next one, because the principle of the next number is valid only in the domain of natural numbers or integers". In the critical tasks, where their explanations clearly indicated a radical change in their thinking about numbers, such as "it is not possible to define the closest number, because it is a limit, where you can always find numbers that are closer", they showed a reduced level of certainty in their estimations compared to the level they showed in more familiar tasks. The reduced certainty of these students suggests the novelty of their thinking and the radical nature of the change experienced.

4.2. Metacognitive strategies as a precondition for grasping the conflicting notions

We tested the effect of explicitly teaching the density of numbers on the number line at the beginning of the students' first course in calculus in a quasi-experimental design (Merenluoto, 2003) with an experimental group ($n = 16$, mean age 16.9 years) and a control group ($n = 24$, mean age 17.2 years). At the pre- and post-tests, we used the same questions on the density of numbers on the number line used in our previously mentioned surveys (Merenluoto & Lehtinen, 2002). In the beginning of the experiment, the students in the experimental group were given an exercise on abstracting the density of numbers on the number line with the help of given questions and asked to write about their experience. This was done in order to activate the students' prior knowledge concerning the aspects of numbers in the number line; to make them aware of their own conceptions of numbers; and to be able to define the quality of their possible metacognitive strategies in dealing with the compact nature of numbers.

There was a significant interaction between the grouping and pre- and post-tests in the answers concerning the density of numbers, $F(1, 45) = 11.89$; $p = 0.002$; with an effect size $\eta^{2}=0.238$ and power 0.919). However, in a qualitative analysis of the change between the tests, it was found that almost half of the students in the experimental group stayed on their "no-relevant-perception" path in spite of the intervention. Further, in the analysis of the written descriptions in the abstraction exercise, a dichotomised distinction in the quality of written metacognitive strategies was found between the students who made a clear change in their explanations compared to those who did not. In the abstraction exercise, the students who made hardly any changes to their answers in the post-test, described their thinking about the number line as a static line with small perpendicular cross lines with a number next to them. Although they explained how there is infinity of numbers, they did not express any strategies to deal with the infinity, nor to handle the irrational numbers. In contrast, the students who made a prominent change in their explanations between the tests, or gave a quite high level of answers already in the pre-test, described their thinking about the number line as a dynamic model where it is possible to change the scale or to "browse" the numbers back and forth in order to find the needed ones. They also demonstrated high level strategies of handling irrational numbers as approximations of rational numbers. For example, Tim, still tackling the conflict between infinite and finite, handles it by presenting a powerful mental strategy of browsing the numbers back and forth, and by abstracting a depth dimension for the number line:

> "In order to examine the irrational numbers I zoom out of the number line, take an approximate value of the needed number, and dive back into the depths of the number line"—Tim—(Merenluoto, 2003: p. 290).

The results suggest that the students who were sensitive to the density of numbers on the number line demonstrated clear metacognitive strategies in handling the conflict between natural and rational numbers already at the beginning of the experiment. The other students did not change their explanations in the post-test but based them on discrete numbers even though the density of numbers on the number line was explicitly taught and discussed during the intervention.

4.3. On-line process on experience-of-conflict path

In order to study the cognitive sensitivity and the tolerance of ambiguity on-line, a special test was planned with tasks of sorting and identifying rational numbers and adding their density on the number line. Six student-teachers who had been high achievers in school mathematics were asked to participate in the tape-recorded interviews and think aloud while working with the tasks. During the interview, we succeeded in catching on-line a process of increased sensitivity leading to the experiencing of ambiguity and avoidance. For example, a student teacher, Ann, after several tasks with decimal numbers, such as adding the density of the decimal numbers on the number line, and sorting of fractions, was presented with a clear conflict when asked to find the "next" number after 3/5:

From the think-aloud protocol (Ann):	Actions and interpretations
I don't know if it is so simple, if I'd start to look for it.	
I think that the next number after 3/5 is 4/5, but I cannot tell why…	She uses the rules of discrete numbers when dealing with fractions. The fact that she is not able to explain, why she thinks that way refers to the self-evident and self-explanatory nature of small discrete numbers.
May be, because it works like this.	Then she starts to use a strategy she is very familiar with: she draws small circles to represent the fractions.
This is a whole number and if I divide it into five parts, then this part is 3/5. Then the next would be one bigger—and 5/5 would be a whole number.	Ann draws a circle to represent a whole number and colours the three parts of the circle, and explains the next number using the "add-one-rule" of discrete numbers.
I do not know… I started to think that… if those parts are divided into smaller parts, then you could have even 15 parts and the next would be 4/15.	But then she experiences increasing sensitivity when she notices, that there are several possibilities to divide the sections of the circle.
But you could divide it still into a lot smaller parts…	Pondering, getting more sensitive.
I don't know what the next number would be, because you can always divide it into smaller and smaller parts—then the next would be …	She—in fact, gives the correct answer—but still struggles with her prior knowledge.
I do not know…mumbling… Isn't it then simply the 4/5?	She does not have enough tolerance for ambiguity and relies on the certainty of her prior thinking.

Ann has some intuition of the difficulty of the task from the beginning, but anyway she refers to her intuition of next numbers on the basis of her prior knowledge. The fact that she is not able to explain why she thinks it works, indicates a powerful intuition about discrete numbers, where the knowledge of "next" as one bigger is self-evident. However, by using her fluent strategy to deal with fractions, she draws a circle to represent the number three-fifths. After this she experiences increasing sensitivity when she concretely notices how there are several different possibilities to find a division close to it. Increased sensitivity leads her to express a profound and correct realization: “you can always divide it into smaller and smaller parts”. However, the conflict it created with discrete numbers in her prior knowledge, and with her belief that there is a numeric to every question in mathematics, leads to a feeling of ambiguity, which was difficult to tolerate at that moment. Thus, after pondering for a minute, she gives up and returns to her prior familiar explanation on next numbers. Thus, the *experience-of-conflict* path has a critical phase where the process results in increasing sensitivity leading to a feeling of ambiguity. However, it is also possible that Ann chooses to rely on her prior knowledge in order to control her feeling of reduced certainty.

5. Discussion

In the theoretical model discussed in this article, three possible learning paths were presented with only one of them leading to a radical conceptual change. The critical processes on latter path were the optimal level of prior knowledge related to the critical aspects in the phenomenon, the sensitivity to the novel features in the situation, and the process of tolerating the ambiguity resulting from the experienced conflict. This suggests that in the process of conceptual change, the students are forced to tolerate the ambiguity that comes from newly learned operations and characteristics of objects, while they do not yet fully understand the concepts. The empirical results indicate that this kind of sensitivity and tolerance are significantly related to students’ achievement level in mathematics. Tsamir and Dreyfus (2002) have reported on this kind of process in a case where one 15 years old and gifted student in mathematics faced the counterintuitive nature of actual infinity. The student’s response indicates that he was both interested in mathematics and had a motivational trust that mathematicians have ways to deal with such conflicts.

> “I cannot be sure. I have to learn about infinite sets in order to know how to make such a decision. Or rather how mathematicians reach the decision”. (Tsamir & Dreyfus, 2002: p. 15).

In the two other paths of the model, the conflict is passed unnoticed either because of the over-confidence resulting from the moderate operational understanding or from dependence on familiar operations. In the third case, it was not recognized because of the students’ wide cognitive distance to the concept to be learned leading to the no-relevant-conception path. The empirical evidence indicates that in the efforts to teach conceptual change it is important to consider the cognitive distance between students’ prior knowledge and the new phenomenon to be learned. The results from our teaching experiment suggested further that in order for the students to be able to deal with the conflicting notions, it is mandatory to use teaching methods that support the development of meta-conceptual awareness and the use of metacognitive strategies in dealing with conflicting notions (see also Wiser & Amin, 2001).

Because conceptual change is a slow process, observing a construction or observing the increasing or the decreasing sensitivity might be problematic. It is possible that these kinds of processes occur when students are sitting alone and think hard, or struggle to tackle a difficult problem. Most of our empirical results come from an analysis of something that has already happened. But in the interviews with some of the student-teachers, it was possible to catch the processes of increasing sensitivity and of dealing with the ambiguity on-line. However, more research is still needed to confirm and elaborate the model proposed in this article.

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