CHAPTER 18

ERNA YACKEL AND CHRIS RASMUSSEN

BELIEFS AND NORMS IN THE MATHEMATICS CLASSROOM

Abstract: The central purpose of this chapter is to demonstrate that by coordinating sociological and psychological perspectives we can explain how changes in beliefs might be initiated and fostered in mathematics classrooms. In particular, we examine: 1) the coordination of students' beliefs about mathematical activity and classroom social norms and 2) the coordination of specifically mathematical beliefs and classroom sociomathematical norms. Examples from a university level differential equations class are used for purposes of clarification and illustration.

1. INTRODUCTION

For more than a decade we and our colleagues[1] have collaborated to study students' mathematical learning in the context of the classroom. In the process of doing so, we have developed an interpretive framework (see Table 1) for analyzing classrooms that coordinates both individual (psychological) and collective (sociological) perspectives. In this work we were strongly influenced by Bauersfeld, Krummheuer, and Voigt's (Bauersfeld, 1988; Bauersfeld, Krummheuer, & Voigt, 1988) long standing work in advancing symbolic interactionism[2] as a theoretical framework for investigating mathematics teaching and learning. The central thesis of this chapter is that by coordinating sociological and psychological perspectives it is possible to develop ways to explain how changes in beliefs might be initiated and fostered in mathematics classrooms. The purpose of this chapter is to develop this thesis. In particular, we discuss those aspects of the interpretive framework that relate to student beliefs and the corresponding classroom norms. The beliefs we consider in this chapter are beliefs about one's role, others' roles, and the general nature of mathematical activity in school and specifically mathematical beliefs and values. We use a university level differential equations class as an example to clarify and illustrate these constructs within the framework. The example demonstrates both the normative aspects of the classroom and the corresponding student beliefs. In each of the classrooms we have studied over the past years, from elementary school mathematics to university level differential equations, student mathematical beliefs changed dramatically over the course of the teaching experiment. In this chapter we demonstrate how the theoretical constructs of the interpretive framework can be used to explain this change. The significance of this work is that it begins to address

G. C. Leder, E. Pehkonen, & G. Törner (Eds.), Beliefs: A Hidden Variable in Mathematics Education? 313-330.

a major challenge in working with beliefs, namely, the initiation of changes in students' beliefs about mathematics and mathematics instruction (also see Greer, Verschaffel, & De Corte, this volume; Tsamir & Tirosh, this volume).

2. BACKGROUND

Following on the seminal work of Erlwanger[3] (1973), a number of mathematics educators have argued for the need to consider students' beliefs about mathematics when attempting to make sense of their mathematical behavior. For example, Cobb (1985) demonstrated that the mathematical activity of the young children who participated in an extended teaching experiment could not be accounted for solely in terms of their mathematical conceptions. However, by complementing a conceptual analysis with an analysis of the children's beliefs it was possible to explain the radically different behavior of children to whom similar concepts were attributed. At the same time, Schoenfeld's work with university level students led to similar conclusions (Schoenfeld, 1983).

As early as 1986, Cobb conjectured that mathematics instruction, as a socialization process, influences student beliefs (see also Greer, Verschaffel, & De Corte, this volume). This conjecture, which was based on working with children in one-on-one settings, was confirmed in a classroom teaching experiment[4] we conducted in 1986-87 in one second-grade classroom. As we have reported elsewhere (Cobb, Yackel, & Wood, 1989), student beliefs at the beginning of the school year were compatible with a "school mathematics tradition," but as the year progressed their beliefs became compatible with an "inquiry mathematics tradition"[5]. Initially,

> [T]he teacher's expectations that the children should [attempt to construct their own solutions to problems and] verbalize how they actually interpreted and attempted to solve the instructional activities ran counter to their prior experiences of mathematics instruction in school (Wood, Cobb, & Yackel, 1988). The teacher, therefore, had to exert her authority in order to help the children reconceptualize their beliefs about both their own roles as students and her role as the teacher during mathematics instruction. She and the children initially negotiated obligations and expectations at the beginning of the school year which made possible the subsequent smooth functioning of the classroom. Once established, this mutually constructed network of obligations and expectations constrained classroom social interactions in the course of which the children constructed mathematical meanings (Blumer, 1969). The patterns of discourse served not to transmit knowledge (Mehan, 1979; Voigt, 1985) but to provide opportunities for children to articulate and reflect on their own and others' mathematical activities. (Cobb et al., 1989, p. 126)

As we will explain below, in order to investigate how it was that student beliefs were influenced by the socialization process, we sought to analyze the social (participation) structure of the classroom. The sociological perspective we followed was that of symbolic interactionism because of its compatibility with psychological constructivism (Voigt, 1996; Yackel & Cobb, 1996)[6]. In the same way that attention to student beliefs is not a logical necessity but proves pragmatically useful because it helps to account for aspects of students' mathematical activity that otherwise are not explainable, taking a sociological perspective is not a logical necessity. However, taking such a perspective proves pragmatically useful because doing so provides means to analyze and ultimately explicate aspects of the teaching and learning of mathematics in the classroom setting that otherwise defy explanation.

3. THE INTERPRETIVE FRAMEWORK

In this section we give a brief overview of the constructs in the interpretive framework that are relevant to this chapter. A more extensive discussion of the framework together with clarifying examples can be found in Cobb and Yackel (1996). First, we wish to emphasize that the interpretive framework is not the result of an a priori theoretical analysis but rather grew out of extensive classroom-based research. It evolved from our attempts to make sense of students' learning in the classroom across several yearlong classroom teaching experiments in elementary school mathematics classes. Our initial efforts included considerable attention to unraveling the complexity of the classroom by focusing on the classroom social norms and later on the sociomathematical norms (Cobb et al., 1989; Yackel, Cobb, & Wood, 1991; Yackel & Cobb, 1996). We use the label social norms to refer to regularities in the interaction patterns that regulate social interactions in the classroom. As such, social norms are expressions of the normative expectancies in the classroom. For example, in a classroom the interaction patterns might be indicative of the expectation that students are to explain their thinking to each other. By contrast, sociomathematical norms refer to regularities in the interaction patterns that are specific to mathematics. For example, in our analyses we have noted that classrooms differ with respect to what becomes normative regarding acceptable mathematical explanations. In this case, the relevant expectations are specifically related to the fact that the subject matter is mathematics, as opposed to history, or literature, or some other subject. Consequently, we have chosen to use the label sociomathematical norms to distinguish these norms from general classroom social norms.

Table1. Interpretive Framework for Analyzing Individual and Collective Activity in Classrooms

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about one's role, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Specifically mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

These constructs are sociological in that they refer to the classroom community as a collective group rather than to the individual members of the community. Nevertheless, in attempting to analyze norms, we took the position that there is a reflexive relationship between the individual and the collective[7]. Therefore, our analyses of norms necessarily involved taking account of the corresponding psychological components. As we have noted elsewhere (Cobb, Yackel, & Wood, 1993) we take beliefs to be the psychological correlates of norms. In doing so, we are taking beliefs to be basically cognitive. They are the understandings that an individual uses in appraising a situation. Thus, discussions of norms and discussions of beliefs are intimately intertwined. This interrelationship between beliefs and norms is critical because it provides a means for talking about changes in beliefs. That is, in saying that norms and beliefs are reflexively related we imply that they evolve together as a dynamic system.

Methodologically, both general social norms and sociomathematical norms are inferred by identifying regularities in patterns of social interaction. Thus social norms are identified from the perspective of the observer and indicate an aspect of the social reality of the classroom. However, what becomes normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. For example, a student‘s inferred beliefs about his or her own role in the classroom, others' roles, and the general nature of mathematical activity can be thought of as a summarization of the obligations and expectations attributed to the student across a variety of situations. In this sense beliefs can be thought of as an individual’s understandings of normative expectancies. Social norms can be thought of as taken-as-shared beliefs that constitute a basis for communication and make possible the smooth flow of classroom interactions (Cobb et al., 1993).

4. AN ILLUSTRATIVE EXAMPLE

To illustrate the reflexive relationship between student beliefs and classroom social and sociomathematical norms, we draw on data from a semester-long classroom teaching experiment conducted in a university level differential equations class. The overall goal of this teaching experiment was to investigate university level students’ learning of differential equations in the context of the classroom. We wish to stress that the interpretive framework and the theoretical relationships discussed above were not developed from data taken from this classroom. These theoretical developments resulted from extensive analysis of several yearlong classroom teaching experiments at the elementary school level. However, the differential equations instructor was knowledgeable about research from which the interpretive framework arose and was committed to the goal of developing an inquiry form of instruction[8]. To this end, he gave explicit attention to the constitution of classroom norms that characterize an inquiry mathematics tradition. In particular, the instructor sought to foster the social norms that students are expected to develop personally meaningful solutions, to explain and justify their thinking, to listen to and attempt to make sense of the thinking of others, and to raise questions and challenges when they disagree or do not understand. Within this context, he also sought to promote the sociomathematical norm that explanations should be descriptions of actions on taken-as-shared mathematical objects that are experientially real for the students.

A question of interest to the project research team was the extent to which these social and sociomathematical norms, which had grown out of analysis of elementary school mathematics classrooms, would be equally beneficial at the university level. For this reason, specific attention was given, on a daily basis, to the norms that were operative in the classroom and to their evolution as the semester progressed. Through classroom observations, it was apparent that the norms described above were constituted in the class over the course of the semester. Further, it was apparent that the individual students altered their beliefs about their role, the teacher's role and the nature of mathematical activity. In the following discussion we document these claims. In keeping with the position that beliefs are the cognitive basis that individuals use to interpret situations that arise in the course of social interaction, our methodological approach is to demonstrate an evolution of students' beliefs across the semester by considering the reflexive relationship between beliefs and norms. This approach stands in contrast to approaches that codify student beliefs before and after the instructional period and compare the results.

4.1. The Project Classroom

The project took place in a differential equations class in an American university. The majority of the twelve students in the class[9] were engineering students; the remainder were mathematics majors. The class met for two-hour class sessions twice a week for fifteen weeks. As noted above, the instructor used an inquiry approach to instruction. Further, as part of the overall project, he developed a majority of the instructional activities used in the class[10]. In addition, the class used a reform-oriented textbook[11] for homework problems. Students used a TI-92 graphing calculator with programs specifically designed by members of the project research team to foster their development of a conceptual basis for slope fields and phase portraits.

In addition to regular homework assignments, students submitted weekly electronic journals in which they reflected on their mathematical activity during the prior week. In some instances specific journal prompts were given. For example, the first journal prompt was to "describe at least one idea from the previous week that was most confusing to you and one idea that was clearest to you". Each student also prepared a course portfolio, which was handed in on two occasions, the day of the first exam and at the final exam. The purpose of the portfolio was for students to synthesize their learning across a number of weeks. To prepare a portfolio a student needed to select entries that reflected important instances of learning and, for each entry, write a rationale explaining the insights gained through the instructional activity represented by the entry. An unanticipated benefit of the electronic journals and portfolios was that they provided additional information about students' evolving beliefs.

The data that form the basis for this chapter come from video recordings that were made of every class session and from the students' electronic journals and portfolios. Throughout the teaching experiment the project team met on an ongoing basis to discuss a variety of issues, including classroom norms and students' beliefs. However, no analyses were conduced until after the semester was completed. At that time two members of the research team set out to document the classroom norms and how they were constituted. For this purpose, our field notes and the video recordings were scrutinized for evidence relating specifically to norms and beliefs, with special attention to the first several weeks of class when the negotiation of norms was initiated. We then identified exemplary episodes from our field notes and videotape review notes. Examples were selected on the basis of their clarity for presentation using verbal material only. These examples were then transcribed. As noted above, an unanticipated benefit of the electronic journals and portfolios was that they provided information about students' beliefs. Consequently, these materials were scrutinized to identify evidence related to the students' initial or evolving beliefs. Again, illustrative examples were selected for use in this chapter based on their clarity. This chapter presents the results of our analysis using the selected episodes and examples as the basis for discussion.

4.2. Coordinating Classroom Social Norms and Beliefs about the Nature of Mathematical Activity

4.2.1. Beliefs

Encountering an inquiry approach to mathematics instruction was a novel experience for the students in the differential equations class since all of them had presumably experienced only the school mathematics tradition in their prior grade K-12 and university mathematics instruction. Thus, at the beginning of the semester their classroom mathematical beliefs were based on expectations that the students' role in class is to follow instructions and to solve problems in the way the instructor and/or textbook demonstrate. Similarly, the instructor's role is to explain and demonstrate procedures for the students to follow. These expectations simultaneously clarify the beliefs students held about the nature of mathematical activity in the classroom. Evidence for these claims about initial student beliefs comes primarily from two sources. First, the interaction patterns in the classroom and second, student comments in their electronic journals. In the paragraphs below we give examples of comments from student journals as a means of documenting their beliefs.

To provide a background for making sense of the students' electronic journals we briefly describe the initial class activity of the first class session. The students' task was to attempt to make sense of the spread of an infectious disease. After a brief discussion of the scenario, the students were to sketch graphs of the susceptible population, the infected population, and the recovered populations over time. They had no quantitative information available to them and no procedural solution method to follow. Thus, the initial classroom activity confronted the students' beliefs about what constitutes mathematical activity, about their role, and about the teacher's role. Subsequent activities were of a similar nature. In fact, all instructional activities throughout the semester were designed to focus on sense making and required explaining and justifying one's thinking and mathematical activity.

In the early weeks of the semester a number of students made explicit comments in their electronic journals about the discrepancy between their (prior) beliefs and the expectations in this class and/or their attempts to cope with the shift in expectations. For example, in the first week's journal one student remarked:

> The only thing that I find sort of confusing is the fact that there may not be an exact answer or answers to a specific problem. There are so many variables in some of these problems that the answers that are obtained (if obtained) can only be used to make educated guesses. I'm used to thinking of math as an "exact science" where there is always an exact answer or answers to a problem.

We interpret this comment to mean that the student's belief that mathematics should always yield one exact answer was being challenged. Another student articulated his earlier belief about the teacher's and the students' roles by saying,

> I'm still getting used to the format. I'm more used to the teacher saying everything and not letting the students really have a "voice".

That he is struggling with these new roles is evident from his later remark in the same journal entry,

> You never said exactly how you wanted the homework done.

Another student commented explicitly about the expectations regarding explanation. He wrote in the journal that accompanied his second homework assignment that he was disappointed in his homework score. He had spent many hours completing the assignment yet he earned only half of the possible points. He wrote the following:

> [M]ost of the points lost were due to my failure to explain how I reached my answers. I thought a clear, systematic approach to the math calculations would be sufficient to explain my thought process. I now have a better understanding of the expectations.

This student is articulating that he formerly believed that in mathematics explanation is not required.

Yet another student was clearly struggling with the dissonance between his beliefs about what mathematics instruction should be and his experiences in the first few class sessions. In his second journal entry, he wrote,

> ... the class is interesting, but the problem is that I'm not learning much. ... I won't deny that I enjoy the open discussion that our class has, but let's face it, we aren't learning much. It would be more practical for us to do some examples using the concepts in class.

Some weeks later, in the portfolio he handed in at the first exam, this same student included the following rationale statement for one of his portfolio entries.

> This handout was chosen because it was one of the initial problems that made me think in a way that I was only exposed to in classes based on philosophy. The open discussion in class at this time was foreign to me, especially in a math class. As you may have guessed, we were not sure what to write or what to answer because we were never asked these types of things before. So basicly [sic], I think that this piece was important because it was the introduction of obscure thinking used in this class.

We take this to mean that while the student had, by this time, gained some understanding of the expectations in this class, they were still counter to what he believed mathematics instruction should be. For him, this class was like philosophy.

Yet another student who had taken the same course previously on another of the university's campuses, with unsuccessful results, was able to clearly articulate the differences in the expectations between the two classes. Although he did not indicate what his beliefs are, he linked his instructional preference to these expectations.

> I like the way the class and the book concentrate on practical applications and explanations for differential equations. As you may have noticed from my info [information] card, I have taken this class before at________[another campus of the same university] and it was much different. We spent a lot of time trying to memorize all the techniques to solve the equations and learned very few practical ideas. I was lost and disinterested 15 minutes into the first class session. I can honestly say I think I've learned more about differential equations in the first two weeks here than I did in the whole semester there.

This student explicitly linked expectations of memorizing techniques to not learning much about differential equations and expectations of explanation to substantive learning. Other students were in the position of making comparisons of this class to other mathematics classes in their prior experience. This student had the advantage of being able to compare two versions of the "same" class. As a result, he could be much more explicit in stating his views.

The above selections, which are only samples from the entire class, demonstrate that, in general, students' beliefs upon entering the class were consistent with the expectations that mathematics consists of using prescribed rules and procedures to find exact answers to problems (cf. Kloosterman, this volume; Tsamir & Tirosh, this volume). Further, the students' comments indicate that these beliefs were being confronted by the instructional approach they were experiencing in the differential equations class. As we have shown elsewhere (Yackel, 1995a), it is the situation as it is interactively constituted as a social event rather than the social setting per se that is critical in influencing the nature of students' mathematical activity and beliefs.

4.2.2. Classroom Social Norms

The differing expectations of the students and the instructor led to situations of explicit negotiation. For example, on the second day of class the instructor began with a brief statement of the expectations regarding classroom participation. He concluded his remarks by saying, "We had some nice examples of that from Shawn and Natasha last time". We know from prior analysis that one of the ways a teacher can initiate the renegotiation of expectations and obligations is through explicit discussions such as this (Cobb et al., 1989). However, such discussions by themselves are insufficient for establishing a classroom in which routines are regulated by those expectations. The students and instructor must come to act in accordance with the expectations. In this case, the next twenty minutes of the class was devoted to a dual agenda. On one hand, the class was engaged in a mathematical discussion about the rationale behind the rate of change equations they had used in the prior class session to model the spread of an infectious disease. On the other hand, the instructor gave explicit attention throughout to the negotiation of social norms compatible with the expectations for inquiry mathematics listed above. In fact, the majority of the instructor's remarks throughout the episode were (explicitly or implicitly) directed toward the expectations. Only a few of his remarks were explanations related to the mathematical content.

Episode 1
Excerpts from the twenty-minute episode are included here.

Instructor: Just to sort of recap, last time we were dealing with the spread of a virus. We had [the] elementary school population in Chicago where we had students who were either susceptible to the disease, who were recovered, or who were infected. And we talked about one differential equation. That [was] $dR/dt = (1/14)I$. Anyone remember why it was one fourteenth? How many people remember? Shawn, why was it 1/14?
Shawn: Fourteen days from the time you got cured, from the time you got it to the time its over.

In keeping with traditional instruction, Shawn might expect the instructor to evaluate his response as correct or incorrect and then initiate a different question. However, the instructor pursues the same question further. He calls for additional explanation, in particular, he asks how it "makes sense". In doing so, he indicates that students' responses should explain their individual thinking and further, that mathematical thinking is about sense making.

Instructor: Okay, can you explain to us then why it was 1/14 times I? How did that sort of make sense as a way to express the change in the recovered population?
Shawn: That it's constant.
Instructor: Say that again.
Shawn: That it's constant. The same amount of number of people for each stage.
Instructor: What do the rest of the people think about that?

With this question the instructor initiates another shift. His question indicates that he expects others to be actively engaged in the discussion. They are to listen to the exchange he and Shawn are having and are to develop their own interpretations about Shawn's response. An implicit expectation is that each student is developing his or her own response to the question even though it was specifically addressed to Shawn initially. Another student offers his thinking.

Jerry: Each day, there's from day zero to day one (inaudible) from day 14 to day 15, you would see 1/14 of that population recover. And every day thereafter.
Instructor: (To Shawn.) Is that similar to what you were thinking?

Here again, the instructor does not follow the traditional initiation-response-evaluate pattern (Mehan, 1979). Instead of evaluating Jerry's remark, the instructor indicates that he expects students to listen to and attempt to make sense of each other's contributions. The instructor is attempting to initiate a genuine dialogue between the students. His intention is that they communicate with each other, not only with and through him.

Shawn: Kinda. Yeah
Instructor: Kinda. What about the explanation here? Did everyone understand what was said?

Our analysis to this point shows that the instructor is attempting to influence the interpretations the students make of how to engage in the discussion. From this perspective, it might seem that the instructor is the only one in the classroom who contributes to the renegotiation of social norms. However, norms for social interaction are interactively constituted as individuals participate in interaction (Yackel & Cobb, 1996). In this case, as the episode continues students contribute their part to the negotiation of the social norms by increasingly acting in accordance with the expectations. As the discussion progressed, students not only responded to the instructor's questions, they initiated comments of their own that showed that they were beginning to change their understandings of the classroom participation structure. Greg's remarks, as the dialogue continues, are an illustration.

Greg: I didn't quite understand what he said.
Instructor: What was that?
Greg: I didn't quite understand what he said.
Instructor: So maybe you could rephrase it or say it a little bit louder so people can hear how you're thinking about it.
Jerry: If every day, we took all the days, we say—, day zero to day one, day one to two, through day 13 to 14, you had population infected, when you got to day 15 we made the assumption that 1/14 of the population was recovered and my understanding is that on day 16 its another 1/14 recovered. It's an assumption that we made.
Instructor: Anyone want to add to that explanation? Expand on it a little bit? Maybe you still have questions about it.

With these remarks, the instructor continues to emphasize the expectations regarding sense making and explanation. Apparently, if you don't understand, even after extended discussion, you are expected to ask for further elaboration. By asking for further clarification, Greg contributes to the constitution of this expectation as normative in this class.

Greg: Well, how I understand it is that ... what I don't understand, what I was asking about, whether the—because, initially we said it was a fixed population, whether the fixed population will have some part to play in the formula because that's what I don't understand—if we said (1/14)I, or what, the fixed population or—
Instructor: A good question, I think I understood what you were saying. Earlier I heard you saying 1/14 of the population. And you kind of said the same thing here. Well, this (points to the differential equation on the chalkboard) is referring to 1/14 of what population?
Jerry: The infected population.

After further discussion involving Jerry, Shawn, and the instructor, another student initiates a question about the assumptions underlying the use of 1/14 as the scalar multiple of I.

Alicia: Could you have [just as] easily assumed that dR/dt is 1/15 or 1/16? Is it just the assumption that there is a correlation between the number of days that the disease will run to [sic] the denominator?

An extended discussion ensued in which Alicia, the instructor, Jerry and Shawn continued to make sense of the relevance of 1/14 in the differential equation of interest.

In the above dialogue we see Greg and Alicia acting in accordance with the expectation that classroom activity is about making sense—making sense of the questions posed by the instructor and others, making sense of the explanations offered as part of the discussion, and making sense of the problem scenario. We also see Jerry and Alicia (in the extended discussion not included here) acting in accordance with the expectation of explaining one's thinking to others. We find it encouraging that as early as this point in the second class session students were beginning to act in accordance with the expectations that the instructor was attempting to initiate just a few moments earlier[12]. We maintain that such actions on the part of students are not simply mindless reactions to the instructor's initiations. Rather, these actions indicate the students' interpretations of the instructor's intentions. Smith (1978) has said, "a willingness to act and ... the assumption of some risk and responsibility for action in relation to a belief represent essential indices of actual believing" (p. 24). Accordingly, we would say that as Greg, Alicia, and Jerry act in accordance with these "new" expectations, their beliefs about their role, the instructor's role, and the nature of classroom mathematical activity are evolving. In this case, not only did these students play a critical role in the interactive constitution of the expectations for the classroom, in doing so, they initiated a shift in their individual beliefs about the classroom participation structure. For his part, by maintaining a focus on the negotiation of expectations, the instructor made it possible for the students to reorganize their beliefs in a way that was compatible with the expectations he was attempting to foster.

The effectiveness of the renegotiation of social norms is indicated by considering classroom interactions that became typical later in the semester. As an example, consider the following episode, which occurred toward the end of the semester.

Episode 2

After two students in the class explained how they determined that a particular phase portrait would not have two saddles next to each other, Dave spontaneously added to the discussion with this remark.

Dave: The way I thought about it at first, to make me think that all the points weren't saddles, is that if the next one was a saddle—see how [Bill] has got the one line coming in towards [referring to the phase portrait that Bill had drawn on the blackboard]. Well, if the next one was like that, then you would have to have another point in between those two equilibrium points, like separating, like a source or something. So that's how I started thinking about it. So then $3\pi/2$ might be a source or maybe a saddle point with opposite directions.

Dave's remark elicits the following response from Bill.

Bill: So it's like, you're saying that if there is a saddle, there has to be a source. If there is a sink or a saddle you have to have a, like in this case right here, you would have to have a source in between the saddles in order for it to really make sense.

Shortly thereafter, Bill relates Dave's explanation to their earlier study of autonomous first order differential equations.

Bill: If you draw the phase line with, like, two sinks, one on top of the other, then you would have to have a source between them.

These spontaneous remarks made by Dave and Bill indicate that they have taken seriously the obligations of developing personally-meaningful solutions, of listening to and attempting to make sense of the thinking of others, and of offering explanations and justifications of their mathematical thinking. In the process of acting in accordance with these expectations they are demonstrating their beliefs about their roles and about the nature of classroom mathematical activity. Furthermore, in acting in accordance with the expectations they are simultaneously contributing to their ongoing constitution. In this way, the normative patterns of interaction serve to sustain the expectations and obligations on which they are based and thus to sustain individual participants' beliefs about their role and about what constitutes mathematical activity in this classroom.

4.3. Coordinating Sociomathematical Norms and Specifically Mathematical Beliefs

Earlier we distinguished between classroom social norms and sociomathematical norms as follows. Social norms are regularities in interaction patterns that regulate the social interactions in the classroom. By contrast, sociomathematical norms refer to regularities in the interaction patterns that relate specifically to the fact that the class is a mathematics class. To the extent that interactions involve interpretations or appraisals of a situation, what becomes normative is constrained by the current beliefs of the classroom participants. In the previous section we discussed the beliefs and norms that relate to the participation structure in the class. In this section, we discuss those beliefs that constrain the mathematical aspects of interactions, called specifically mathematical beliefs, and sociomathematical norms[13].

In contrast to social norms, it took much longer to achieve stability with respect to the sociomathematical norms that characterize inquiry mathematics. In this section, we limit the discussion to the sociomathematical norm of what constitutes an acceptable mathematical explanation and related beliefs. In particular, we discuss two aspects of acceptability with respect to mathematical explanation. The first relates to the communication aspect of explanation and the second to the specific expectations that had been established within the classroom in question. In general, an explanation is an individual's attempt to clarify for others aspects of one's thinking that one judges might not be clear (Cobb, Wood, Yackel, & McNeal, 1992). That is, explanation is a form of communication. As such, an explanation is acceptable when it serves a clarifying function. This is one aspect regarding acceptability of explanation. Acceptability, however, cannot be judged apart from those who are attempting to make sense of the explanation. This means that the notion of what constitutes an acceptable mathematical explanation is confounded by the students' developing understanding of the mathematical concepts in the course and the instructor's understanding of students' mathematics. This is where the second aspect of acceptability comes to the fore. For an explanation to be acceptable it has to meet the requirements that have been established through interaction by the participants in the classroom. In general, in mathematics classrooms that follow the inquiry tradition, explanations have to be about actions on mathematical objects that are experientially real for the students. By this we mean that explanations have to be about students' mathematical activity with entities that are part of their mathematical worlds. Descriptions of procedures are typically not acceptable. This is the second aspect of acceptability.

In the following paragraphs we first discuss the general clarifying function of explanation. Next, we discuss the specifically mathematical aspects regarding the acceptability of an explanation. The interactive constitution of what constitutes an acceptable mathematical explanation is closely linked to that of the social norm that explanations are to be given. In this regard, we point to one of the instructor's comments in Episode 1. In response to Greg's remark, "I didn't quite understand what he said," the instructor replied, "So maybe you could rephrase it or say it a little bit louder so people can hear how you're thinking about it". Here the instructor's suggestion to rephrase hints at the communicative function of explanation. If the earlier remark was simply not heard, say it louder. However, if the earlier remark did not serve a clarifying function for the listeners, rephrase it.

There is considerable evidence that by the fourth electronic journal assignment the students were beginning to understand the clarifying function of explanation. In this journal students were asked to explain their understanding of the Existence and Uniqueness Theorem for solutions to differential equations. Many students wrote comments to the effect that their explanations were inadequate. For example, one student ended his journal with this comment, "This is the best way I know to explain it which I know is very lacking". Another wrote, "I am not completely sure I understand this point so I wouldn't try to explain it to someone unless they had some feedback as to what they think it is". We take these comments as indications of the students' belief that the purpose of explanations is to communicate; explanations should clarify one's thinking for others.

The second aspect of adequacy of explanation is specifically mathematical. At the beginning of the semester at least some of the students believed that procedures in the form of calculations were acceptable as explanations. One of the student journal entries we included earlier exemplifies this belief. This was the second journal of the semester.

> [M]ost of the [homework] points lost were due to my failure to explain how I reached my answers. I thought a clear, systematic approach to the math calculations would be sufficient to explain my thought process. I now have a better understanding of the expectations.

Previously we argued that this journal entry shows that the student initially believed that in mathematics explanation is not required. In making that statement, we were using explanation in the sense of providing some insights into one's thinking. From that perspective, the student was not giving an explanation at all. However, from the student's perspective he was giving what he thought counted as an explanation. In terms of what constitutes acceptability, we can say that the student initially believed that systematic calculations constituted acceptable explanations. From the instructor's perspective systematic calculations do not necessarily signify actions on mathematical objects that are meaningful to the student. They may simply be manipulations of symbols.

Earlier we wrote that in mathematics classrooms that follow an inquiry mathematics tradition explanations should describe actions on mathematical objects that are experientially real for the students. We refer the reader back to Episode 2 in an earlier part of this chapter as evidence that this sociomathematical norm was constituted in the differential equations class. In that episode both Dave and Bill talk about sinks, saddles, and sources as entities within their mathematical realities that they can identify and locate. In doing so, they not only act in accordance with this sociomathematical norm, they also demonstrate that their underlying beliefs about mathematical explanations are compatible with that norm.

In this differential equations class a more specific sociomathematical norm regarding acceptable explanation was constituted in the case of first order differential equations. In this case, to be acceptable, explanations had to be grounded in an interpretation of the rates of change as expressed by the differential equation(s). To clarify what we mean by this and to illustrate the constitution of sociomathematical norms, we include the following episode that occurred during the fourth week of the semester.

4.3.1. Episode 3

The dialogue is taken from a whole class discussion of the differential equation dP/dt = 0.5P(1-P/8)(P/3 - 1) which models the rate of change in the population of fox squirrels in the Rocky Mountains. The task posed to the students was to figure out what interpretations might be given to the numerical values 0.5, 8, and 3. Students had discussed the problem in their small groups prior to the whole class discussion.

Instructor: Okay, so Jerry says that if the population gets above 8 they [the fox squirrels] are going to start dying. Tell us why you made that conclusion.

Jerry: Because some number greater than 8 over 8 is going to yield some number greater than one, which 1 minus something greater than 1 is going to give you a negative number and so something times a negative number is going to give you a negative number, so your slope is going to be negative.

Jerry’s explanation is completely calculational in nature. At no time does he refer to what the quantities represent or to how they might be interpreted in the underlying scenario. We take this as evidence that Jerry believes descriptions of procedures such as this constitute adequate explanations. The instructor uses this as an opportunity to initiate a shift in the orientation of the discussion—away from a calculational orientation and towards a conceptual orientation (Thompson, Philipp, Thompson, & Boyd, 1994).

Instructor: So if P is bigger than 8, like you said, maybe 8 million or 8 thousand fox squirrels, then this term here is negative, like you said, right?

Jerry: Mmm hmm.

Instructor: And so what does that mean for us? That means what? If this term is negative, that doesn't tell us anything in itself in relation to the differential equation.

Jerry: The change is negative.

Instructor: Okay, so this part here is negative, is this part negative or positive?
Sylvia: Positive.
Instructor: All right, if P is bigger than 8, certainly 8/3 - 1 is positive and so this is positive, and this [0.5P] is positive, so the rate of change, dP/dt, is negative. So that means dP/dt is negative, which means what?
Greg: The population [is] reducing.
Instructor: They're reducing, good. So the rate of change is negative that means the population, the number of fox squirrels is getting smaller. The population is decreasing. So the number of squirrels (i.e., P(t)) is decreasing.

In the preceding dialogue the instructor repeatedly asks for the meaning of the quantities involved, suggesting that none of the explanations given thus far have been adequate in addressing that issue. Eventually he explicitly tells the class his own interpretation when he says, "So the rate of change is negative *means* (emphasis added) the number of fox squirrels is getting smaller. The population is decreasing". In doing so, the instructor not only confirms Greg's response but also gives the students an opportunity to reorganize their beliefs about the criteria for meaningful explanations. As the discussion progressed, the instructor asked Dave how his group thought about the problem.

Dave: Well, pretty much, kind of, the same as what Jerry was saying but just the opposite. In this case, it says the fertile adults have to be able to find other fertile adults to be able to increase. Well, if they don't, then the rate of change of that is going to be negative which makes everything else negative, so it's decreasing.

Dave's response indicates the effectiveness of the instructor's intervention. Although his response is not at all calculational (as Jerry's initial explanation was), Dave prefaces his response with, "pretty much the same as what Jerry was saying". In saying this, he indicates that he now understands that explanations are to be about rates of change and their significance within the scenario.

The above example illustrates what we mean by the sociomathematical norm that explanations had to be grounded in an interpretation of rates of change as expressed by first order differential equations. It also demonstrates that sociomathematical norms such as this are constituted in interaction. As with social norms, while the instructor typically initiates the negotiation of sociomathematical norms, other participants contribute to their constitution. In the above example, Dave's comment not only exemplified the norm, it contributed to its ongoing constitution. Further, we maintain that as students' actions are increasingly in accordance with the expectations for explanation, they demonstrate their evolving beliefs about what constitutes an acceptable mathematical explanation in this class.

5. CONCLUSION

The chapters in this book show that there are many different ways to investigate student beliefs about mathematics. Our purpose in this chapter has been to demonstrate that by coordinating sociological and psychological perspectives it is possible to explain how changes in beliefs might be initiated and fostered in mathematics classrooms. In particular, we have attempted to demonstrate how classroom social and sociomathematical norms and individual beliefs evolve together as a dynamic system. We have demonstrated that students' beliefs about their role and others' roles in the classroom and about the general nature of mathematical activity evolve in tandem with the social norms that students are expected to develop personally-meaningful solutions, to explain and justify their thinking, to listen to and attempt to make sense of the thinking of others, and to raise questions and challenges when they disagree or do not understand. Similarly, we have demonstrated that students' beliefs about what constitutes mathematical explanation evolved in tandem with the sociomathematical norm that, in general, explanations should signify actions on mathematical objects that are meaningful to the students and, in particular, in the differential equations class explanations had to be grounded in an interpretation of the rates of change as expressed by the differential equation(s). In doing so, we have shown that changes in beliefs and negotiation of classroom norms are inextricably linked.

By coordinating perspectives, we give primacy neither to the social nor the psychological. Rather, we maintain that each provides a backdrop against which to consider the other. Our purpose has been to clarify that as classroom norms are renegotiated, there is a concomitant evolution of individual beliefs. Verschaffel, Greer, and De Corte (1999) have noted that it is generally assumed that students' beliefs about mathematical activity develop "implicitly, gradually, and tacitly through being immersed in the culture of the mathematics classroom" (p. 142). We agree but would argue that one way to give explicit attention to student beliefs in the mathematics classroom is to be deliberate about initiating the negotiation of classroom norms.

6. ACKNOWLEDGEMENTS

The research reported in this chapter was supported in part by the National Science Foundation under grant No. REC-9875388. The opinions expressed do not necessarily reflect the views of the foundation.

7. NOTES

[1] The colleagues with whom we have worked over the past decade or more include Paul Cobb, Koeno Gravemeijer, Terry Wood, Grayson Wheatley and Diana Underwood. Paul Cobb and Erna Yackel developed the interpretive framework that forms the basis for this chapter.

[2] A detailed discussion of the nature of symbolic interactionism and its methodological position can be found in Blumer (1969).

[3] In 1973 Erlwanger published a study in which he investigated the beliefs of sixth-grade student, Benny. Benny was making good progress in mathematics using an approach to instruction that was based on individualized instructional technology. Erlwanger found upon talking with Benny that, despite his good progress, he had understood incorrectly some aspects of his work. In particular, he had many misconceptions about decimals and fractions and how to operate with them. In addition, Erlwanger found that Benny had developed learning habits and views about mathematics that would impede his progress in the future. Erlwanger concluded that the type of instruction Benny received "tends to develop in the pupil an inflexible rule-oriented attitude toward mathematics, in which rules that conflict with intuition are considered 'magical' and the quest for answers 'a wild goose chase" (p.25).

[4] The classroom teaching experiment methodology referred to in this chapter was developed by Cobb, Wood, and Yackel (1991) to extend to the classroom setting the type of one-on-one teaching experiment

that has been used extensively by Steffe and colleagues to investigate children's mathematical activity and learning (Cobb & Steffe, 1983). For a detailed discussion of the classroom teaching experiment methodology see Cobb (1999) and Yackel (1995b).

[5] In using the labels "school mathematics tradition" and "inquiry tradition" we are following Richards (1991) who characterizes the "school mathematics tradition" as one in which students are treated as passive recipients of information and the "inquiry mathematics tradition" as one designed to teach students the language of mathematical literacy. Richards likens the discourse in the school mathematics tradition to "a type of 'number talk' that is driven by computation" (p. 16). By contrast, discourse in the inquiry tradition involves discussions in which individuals interact to attempt to explain and justify their mathematical activity to one another (cf. Thompson, Philipp, Thompson, & Boyd's discussion of calculational versus conceptual orientation, 1994).

[6] Voigt (1996) argues that of the various theoretical approaches to social interaction, the symbolic interactionist approach is particularly useful when studying children's learning in inquiry mathematics classrooms because it emphasizes the individual's sense-making processes as well as the social processes. Rather than attempting to deduce an individual's learning from social and cultural processes or vice versa, symbolic interactionism sees individuals as developing their personal understandings as they participate in negotiating classroom norms, including those that are specific to mathematics.

[7] Such a position is open to empirical verification. Our results from prior analyses confirmed that this is a viable position.

[8] Chris Rasmussen was the course instructor. Erna Yackel attended every class session. They, together with mathematics educator Karen King, formed the project team for the classroom teaching experiment.

[9] This small class size, which is much smaller than the typical differential equations class in an American University, made the class ideal for conducting a teaching experiment. It was possible for the members of the research team to have a rather intimate knowledge of each class member's conceptual understandings as the semester progressed. This information contributed significantly to the instructor's ability to develop appropriate instructional activities and strategies. Subsequently, Rasmussen has successfully used the instructional approach and activities first developed in this class with class sizes of thirty or more students.

[10] Realistic Mathematics Education instructional design theory, developed at the Freudenthal Institute, The Netherlands (Gravemeijer, 1994) informed the instructional design.

[11] The textbook emphasized graphical and other qualitative approaches along with the use of technology as means to solve problems involving differential equations. By contrast, traditional textbooks emphasize a variety of analytic methods to solve various types of differential equations.

[12] One of the ways we can judge the usefulness of theoretical ideas is the extent to which they are applicable in practice. In this case, we have repeatedly experienced in our own mathematics classrooms the rapidity with which we can initiate the renegotiation of social norms for classroom participation within the first one or two class sessions.

[13] In contrast to the previous section where we first documented students' beliefs and then analyzed the constitution of the corresponding social norms, in this section we discuss how students' specifically mathematical beliefs and the corresponding sociomathematical norms evolved together as a dynamic system. Therefore we will not separate the documentation of students' mathematical beliefs from the analysis of sociomathematical norms.

8. REFERENCES

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In T. Cooney & D. Grouws (Eds.), *Effective mathematics teaching* (pp. 27-46). Reston, VA: National Council of Teachers of Mathematics/Erlbaum.

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), *Foundations and methodology of the discipline of mathematics education* (pp. 174-188). Antwerp: Proceedings of the Theory of Mathematics Education Conference.

Blumer, H. (1969). *Symbolic interactionism.* Englewood Cliffs, NJ: Prentice-Hall.

Cobb, P. (1985). Two children's anticipations, beliefs, and motivations. *Educational Studies in Mathematics, 16,* 111-125.

Cobb, P. (1986). Contexts, goals, beliefs, and learning mathematics. *For the Learning of Mathematics, 6*(2), 2-9.

Cobb, P. (1999). Conducting teaching experiments in collaboration with teachers. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 307-333). Mahwah, NJ: Lawrence Erlbaum Associates.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. *Journal for Research in Mathematics Education, 14,* 83-94.

Cobb, P., Wood, T., & Yackel, E. (1991). Learning through problem solving: A constructivist approach to second grade mathematics. In E. von Glasersfeld (Ed.), *Constructivism in mathematics education* (pp. 157-176). Dordrecht: Kluwer.

Cobb, P., Wood, T., Yackel, E., & McNeal (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29,* 573-604.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist, 31* (3/4), 175-190.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117-148). New York: Springer-Verlag.

Cobb, P., Yackel, E., & Wood, T. (1993). Learning mathematics: Multiple perspectives, theoretical orientation. In T. Wood, P. Cobb, E. Yackel, & D. Dillon (Eds.), *Rethinking elementary school mathematics: Insights and issues. Journal for Research in Mathematics Education Monograph Series, Number 6,* (pp. 21-32). Reston, VA: National Council of Teachers of Mathematics.

Erlwanger, S. H. (1973). Benny's concept of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1,* 7-26.

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25,* 443-471.

Mehan, H. (1979). *Learning lessons: Social organization in the classroom.* Cambridge, MA: Harvard University Press.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Constructivism in mathematics education* (pp. 13-52). Dordrecht, The Netherlands: Kluwer.

Schoenfeld, A. H. (1983, April). *Theoretical and pragmatic issues in the design of mathematical "problem-solving".* Paper presented at the Annual Meeting of the American Educational Research Association, Montreal.

Smith, J. E. (1978). *Purpose and thought: The meaning of pragmatism.* Chicago: University of Chicago Press.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. (1994). Calculational and conceptual orientations in teaching mathematics. In D. Achiele & A. F. Coxford (Eds.), *Professional development of teachers of mathematics* (pp. 79-92). Reston, VA: National Council of Teachers of Mathematics.

Verschaffel, L, Greer, B., & De Corte, E. (1999). Pupils' beliefs about the role of real-world knowledge in mathematical modelling of school arithmetic word problems. In E. Pehkonen & G. Törner (Eds.), *Mathematical beliefs and their impact on teaching and learning of mathematics: Proceedings of the workshop in Oberwolfach* (pp. 138-145). Duisburg, Germany: Gerhard Mercator Universität.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathématiques,* 6, 69-118.

Voigt, J. (1996). Negotiation of mathematical meaning in classroom practices: Social interaction and learning mathematics. In L. P. Steffe, P. Nesher, P, Cobb. G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 21-50). Mahwah, NJ: Lawrence Erlbaum Associates.

Wood, T., Cobb, P., & Yackel, E. (1988, April). *The influence of change in teacher's beliefs about mathematics instruction on reading instruction.* Paper presented at the annual meeting of the American Educational Research Association, New Orleans.

Yackel, E. (1995a). Children's talk in inquiry mathematics classrooms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 131-162). Hillsdale, NJ: Lawrence Erlbaum Associates.

Yackel, E. (1995b). *The classroom teaching experiment.* Unpublished manuscript, Department of Mathematics, Computer Science, and Statistics, Purdue University Calumet, IN.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27,* 458-477.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22,* 390-408.