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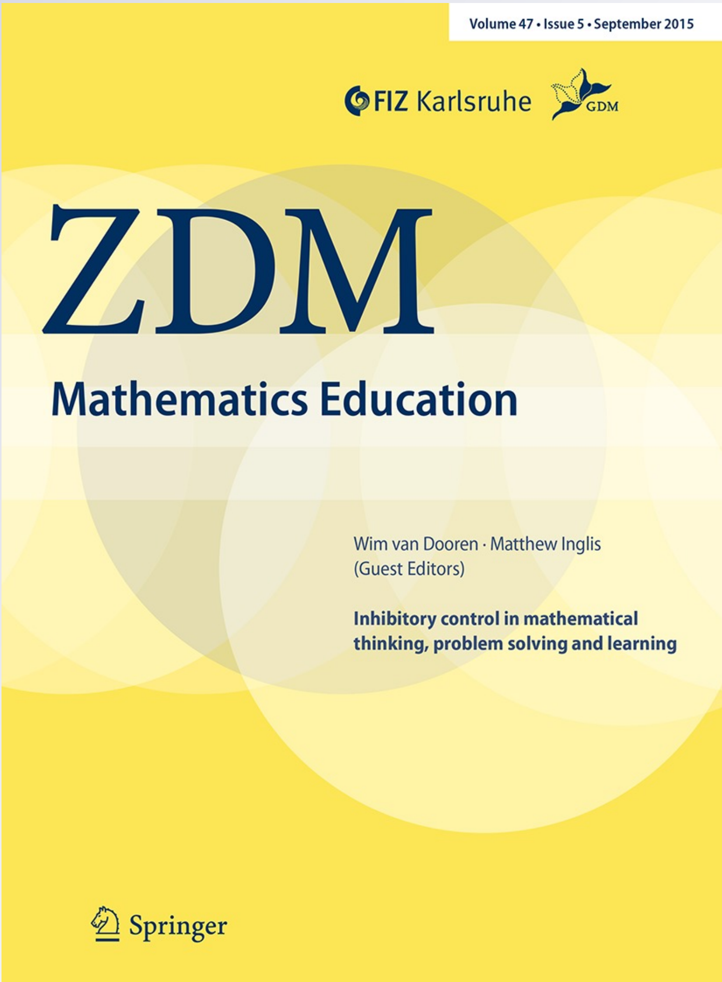

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ORIGINAL ARTICLE

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Abstract This study investigates the hypothesis that there is a natural number bias that influences how students understand the effects of arithmetical operations involving both Arabic numerals and numbers that are represented by symbols for missing numbers. It also investigates whether this bias correlates with other aspects of students' understanding of the number concept beyond natural numbers. Natural number bias has been characterized as the interference of natural number knowledge in reasoning about non-natural numbers. Quantitative data is presented showing that in the case of operations between numbers and missing numbers this bias acts in two main ways. First, it shapes students' anticipations about the expected outcome of each operation, that is, that the result of addition or multiplication "must" be bigger than the initial numbers and the result of subtraction or division "must" be smaller. Second, it causes students to think that missing numbers stand mostly for natural numbers; this tendency would lead students to make decisions about the general results of operations by substituting only natural numbers for the missing number symbols. It is argued that knowledge about operations between natural numbers needs to be inhibited for students to overcome the natural number bias and to reason with numbers beyond the scope of natural numbers.

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1 Introduction

The purpose of this study is to provide quantitative data on the influence of the natural number bias in students' reasoning with arithmetic operations and the way it relates to certain aspects of students' understanding of the number concept. The *natural number bias* characterizes students' tendency to ascribe characteristics of natural numbers to non-natural numbers, a practice that most often results in certain kinds of mistakes on the part of the student (Ni and Zhou 2005). These mistakes appear due to differences between natural and non-natural numbers, which are apparent in notation, the ways of ordering, and the density of their structure, among other reasons; these differences are presented in more detail later on. Recently, the focus of research uses a natural number bias perspective to further investigate the effects of arithmetic operations and more specifically the tendency to anticipate specific results from specific operations, such as larger numbers for addition and multiplication, and smaller numbers for subtraction and division (Vamvakoussi et al. 2012, 2013; Van Hoof et al. 2014).

1.1 The natural number bias phenomenon

The term *whole number bias* was introduced by Ni and Zhou (2005) to characterize children's tendency to use the single-unit counting scheme to interpret instructional data on fractions, a well-known phenomenon in the research literature. This term has since been used interchangeably with the term *natural number bias*, which is often used with the same meaning (DeWolf and Vosniadou 2014; Obersteiner et al. 2013; Vamvakoussi et al. 2012; Van Hoof et al. 2014). Whole numbers are exact numbers, which in a mathematical context are named *integers*, that is, the natural numbers, their negatives, and zero. When the focus is on the issue of the differences between the exact numbers and their fractions, using the term *whole number bias* could be sufficient. However, if the focus is on the interference of natural number knowledge on non-natural numbers, which may include not only the rational numbers but also the negative integers and the real numbers, then the term *natural number bias* may be more appropriate (see also Van Hoof et al. 2014). Therefore the term *natural number bias* is adopted here in order to better describe the conflicted procedures of attributing characteristics and properties of the natural numbers to different kinds of non-natural numbers.

1.2 The natural number bias and the number concept

There is an ongoing debate about the origins of the natural number bias and its relation to the way the number concept is perceived and developed (Rips et al. 2008). However, there seems to be a consensus that cultural privilege, the type of notation, intuitions, and early instruction have all supported the construction of an initial conception for numbers grounded on natural numbers and the act of counting (Gelman 2000; Smith et al. 2005).

This initial conception may undermine students' ideas about *what a number is supposed to look like and how it is supposed to behave* (Smith et al. 2005; Vamvakoussi and Vosniadou 2010). Students often use this early understanding for numbers when reasoning about non-natural numbers by means of projecting their initial knowledge of natural numbers onto non-natural numbers such as fractions and decimal numbers. This has multiple consequences that appear in different mathematical contexts and it is reflected in certain kinds of mistakes that come up. For example, in the case of decimal numbers, students erroneously think that *longer decimals are larger*, for example that 2.367 is larger than 2.6 (Nesher and Peled 1986; Resnick et al. 1989). However, this type of error appears to decrease with age while *shorter is larger* types of responses, a possible result of an intrusion of knowledge about fractions, appear in older children and adults (Stacey and Steinle 1999).

Similarly, it appears that students tend to confound the number of pieces in a partition with the size of each piece (e.g., 1/4 is bigger than 1/3 because 4 is bigger that 3) and to think that *the bigger the numerator and denominator of a fraction, the bigger the fraction* (Hartnett and Gelman 1998; Moss 2005), or to think that *the smaller the whole number components the larger the fraction* (DeWolf and Vosniadou 2014; Resnick et al. 1989; Stafylidou and Vosniadou 2004). Furthermore, the relationship between '1' and a fraction also seems to puzzle students. Some students tend to think that 1 is greater than any fraction while other students think that 1 is less than any fraction (Stafylidou and Vosniadou 2004). The former misconception is likely associated with the part–whole scheme that is most commonly used for introducing fractions to students (Steffe 2002; Thompson and Saldanha 2003); this scheme supports an understanding of fractions as part of a whole and so the fraction always appears smaller than 1. Furthermore, the former misunderstanding could be due to an erroneous overgeneralization in which the property of 1 as the smallest natural number is also applied to rational numbers (see Stafylidou and Vosniadou 2004).

Natural numbers also differ from rational numbers by means of the density of their structure. In contrast to natural numbers, where there is a unique successor and a unique predecessor for each one of them, rational and real numbers are dense in the sense that there are infinitely many numbers between any two rational (or real) numbers. Many research studies have shown that students rely on their initial knowledge of natural numbers and think that numbers in general are discrete, meaning that every number has a unique successor, and that there is no number between two pseudosuccessive numbers, such as 0.5 and 0.6 (Hannula et al. 2006; Merenluoto and Lehtinen 2002; Vamvakoussi and Vosniadou 2010).

Recent findings showed that acquiring a dense structure for rational numbers, meaning to accept that infinitely many numbers lie in any given interval, is not an *all or nothing situation* (Vamvakoussi and Vosniadou 2010). Rather, there are intermediate levels of understanding, where the presupposition of discreteness is present even if it sometimes appears in a more sophisticated way. There is both a mathematically naïve level of understanding, where students respond that there is no number between two pseudosuccessive numbers, and a sophisticated level, where students can accept that there are infinitely many numbers in any interval. In between, there are students whose responses reflect a tendency to apply the property of an *advanced discreteness* (Vamvakoussi and Vosniadou 2010) to the rational number sets, by accepting a finite number of intermediate numbers between two pseudosuccessive numbers that could be named one by one. Students in this category think that between 0.5 and 0.6 there are ten numbers, i.e., 0.51, 0.52, etc., or even one hundred numbers, i.e., 0.511, 0.512, …, 0.521, 0.522,… but not infinitely many numbers that cannot all be named.

1.3 Natural number bias in arithmetic operations

The research considering the natural number bias phenomenon has recently reached the area of arithmetic operations, presenting findings such as students' tendency to think that addition and multiplication between any two numbers always result in a larger number, as well as that subtraction and division always produce smaller numbers, regardless of the kinds of numbers involved (Vamvakoussi et al. 2012, 2013; Van Hoof et al. 2014). It would be plausible to expect that the natural number bias would interfere in this specific way with students' anticipations about the results of arithmetic operations. That is because the result of addition or multiplication between two natural numbers is always a number bigger than the two initial numbers (unless 0 or 1 are involved). Similarly, the result of subtraction or division between two natural numbers is a number smaller than the minuend and the dividend, respectively. This is, however, not necessarily true for certain kinds of non-natural numbers, for which the effects of operations depend on the numbers involved. Specifically, in the case of rational numbers smaller than 1 or negative numbers, students' expectations about the results of multiplication and division in the first case and addition and subtraction in the second case are falsified. For instance, 7: 0.2 is bigger than 7, 8×0.5 is smaller than 8, and $6 + (-2)$ is smaller than 6. This is a well-known phenomenon by every teacher and also in the research literature on students' difficulties with rational numbers, often called by its descriptive name *multiplication makes bigger*.

Back in the 1980s, many colleagues from different perspectives had noted that students tend to overgeneralize the results of operations between natural numbers to also include rational numbers, a tendency that resulted in the above-mentioned misconception. This appeared both in purely numerical items (Greer 1987, 1989) and in word-problem tasks (Bell et al. 1981; Graeber et al. 1989; Hart 1981), when, for example, students choose an operation between two numbers based on their belief that multiplication always produces a larger number and that division always produces a smaller number (Fischbein et al. 1985; Harel et al. 1994). There is also some evidence showing that students tend to associate *more* with addition and *less* with subtraction when solving word problems (De Corte et al. 1990), which may support the claim that students also hold beliefs such as that *addition always makes bigger* and *subtraction always makes smaller* (Green et al. 2008; Tirosh et al. 2008). However, until recently, very little research had specifically investigated students' intuitions concerning addition and subtraction. Fischbein et al. (1985) interpreted those intuitions to lie in certain primitive, implicit models students hold for each operation, that is, the model of addition as putting together, subtraction as taking away, multiplication as repeated addition, and division as partitive and quotitive division.

Quite recently, these misconceptions were revisited from a broader *natural number bias* perspective, using new methodologies and theoretical frameworks. From such perspective Vamvakoussi and her colleagues (2013) argue that the primitive models as described by Fischbein et al. (1985) are compatible with—and based on—natural number operations, particularly with the characteristic that their effects (i.e., whether the result will be larger or smaller) depend merely on the operation and not on the numbers involved. In their studies, Vamvakoussi and her colleagues (Vamvakoussi et al. 2012, 2013) used a pioneering methodology to investigate intuitive reasoning about all four operations in more depth. They measured adult participants' reaction times as they were reasoning about arithmetic operations with given numbers and literal symbols in algebraic expressions (e.g., $5 + 2x$). The results were in line with the hypothesis that even adults have strong intuitions about the results of each operation due to the natural number bias phenomenon. The same phenomenon appeared in a recent paper-and-pencil study with middle grade students (Van Hoof et al. 2014). However, Vamvakoussi et al. (2012, 2013) only partly interpreted their results as an influence of the natural number bias. The researchers argued that students' tendencies to think, for example, that $5 + 2x$ is always bigger than 5 is due to participants' preconceptions about what the specific operation sign represents. The researchers acknowledge that students may be using a particular strategy, namely, trying specific numbers in order to check the results of the given operations, and that this strategy could also be affected by a predisposition to natural numbers (Vamvakoussi et al. 2013).

This explanation is also supported by evidence showing that students indeed tend to substitute literal symbols of algebraic expressions mostly with natural numbers when they need to determine the value of those expressions (Christou and Vosniadou 2005, 2012). In our early studies we have shown that when presented with algebraic expressions involving operations between numbers and literal symbols (e.g., $4\ g$), students tended to substitute literal symbols mostly with natural numbers in order to decide on the value of the expressions (Christou and Vosniadou 2005, 2012). Additionally, in an interview study with tenth graders, students were asked a series of questions such as whether they think that $5d$ is always bigger than $4/d$. The majority of students claimed that $5d$ is always bigger "because multiplication makes the numbers bigger than division," and many of them supported their standpoint by substituting specific numbers for the literal symbols, which in most cases were natural numbers, despite the hints provided by the interviewer to also try with *other kinds of numbers* (Christou and Vosniadou 2012). In support of this finding, Van Hoof et al.'s (2014) study provided more data from students' individual interviews in which the students explicitly referred to general natural number rules or substituted literal symbols with natural numbers to come to an answer. However, there is still need for quantitative data to support a possible dual effect of the natural number bias on the results of arithmetic operations. This gap will hopefully be filled by the present research study.

1.4 The present study

To sum up, in the present study the natural number bias perspective is taken to investigate students' tendency to anticipate certain results from each arithmetic operation. As mentioned above, the natural number bias shapes students' conceptions about what counts as a number and how numbers are supposed to behave, therefore it would be expected that this bias acts in two main ways when students reason about arithmetic operations with missing numbers. (a) It may lead students to decide on the result of a given operation using their preconceptions about the effect of this operation, as if its result (i.e., whether the result will be larger or smaller than the initial numbers) depended exclusively on the operation and not on the numbers involved. As discussed above, these preconceptions, which may take the form of general rules, have been cultivated by students' experiences with numbers which, for many years—from their early acquaintance with the counting numbers until their first years of instruction—were restricted to include the natural numbers almost exclusively. In other words, students would tend to count on general rules such as *multiplication makes bigger* because that is what natural numbers do. (b) It may prompt students to test the result of the given operation between given and missing numbers by substituting the missing ones only with random natural numbers, generalizing the effect of the operation based on the specific results of this testing. The natural number bias will cause students to prioritize substituting missing numbers with natural numbers because, under this general bias, only the natural numbers count as numbers since this is the dominant category of numbers. Consequently, natural numbers are the first kind of numbers that come to someone's mind, and in some cases are the only ones.

Thus, the main hypothesis of this study is that students' tendency to anticipate certain results from specific arithmetic operations is rooted in the natural number bias phenomenon, which affects both of the strategies they use when reasoning about the results of arithmetic operations, namely their tendency to count on a general law about the result of each operation or the strategy to try specific numbers. In order to test this hypothesis and the more general, underlying issue of the natural number bias phenomenon in arithmetic operations, specific tasks were developed. These tasks aimed at illustrating the way the natural number bias affects each of the two main strategies mentioned above, and at providing quantitative data from students' responses that would reveal the way this bias affects each of the students' strategies to reason about the results of arithmetic operations. The tasks involved arithmetic operations between given and missing numbers and were either *congruent* in that reasoning relying on natural numbers would lead to a correct answer, or *incongruent*, meaning that reasoning relying on the natural number knowledge would lead to an incorrect answer. If the students were to exclusively count on the operation sign making use of a general rule in order to determine the effects of the operations (i.e. bigger result in multiplication, smaller result in division), without caring about the numbers involved, there would be no differences in students' responses on the different sets of given tasks.

A sub-focus of this study was to also test whether students' ways of reasoning about specific arithmetic operations would be related to their understanding of the number concept, which is also affected by the same bias as presented above. In order to test whether there are such relationships, the participants also completed a number of tasks that tested their understanding of rational numbers. The tasks included questions about the ordering of fractions and decimal numbers as well as about the density of the rational number set.

2 Methodology

2.1 Participants

The participants in this study were 189 fifth and sixth graders from two public primary schools in Greece; 104 were boys and 85 were girls; 73 were from the fifth and 116 from the sixth grade. The study targeted students of this age to be as close to fourth grade as possible, when students start to learn about operations with rational numbers, which violates their initial intuitions about the effects of arithmetic operations. However, one of the drawbacks with using students of this age is that they lack knowledge of negative numbers (as they are not introduced until seventh grade), which are the kinds of numbers that violate students' intuitions about the results of addition and subtraction, namely that addition always produces bigger numbers and subtraction always produces smaller numbers [e.g., $4 + (-1) = 3$, $4 - (-1) = 5$]. Another issue with choosing students of this age is that they have not yet been introduced to literal symbols as symbols that stand for numbers. Because of this, in the tasks involving operations between given and missing numbers, the missing ones were represented by missing-number symbols (i.e., "_") instead of literal symbols as students of this age have extensive experience with missing-number symbols when practicing calculations.

Table 1 Overview of experimental items by kind of operation and type (congruent, incongruent)

Tasks	Multiplication	Division	Addition	Subtraction
Congruent with natural numbers	$7 \times \_ = 21$ $3 \times \_ = 9$	$8{:}\ \_ = 2$ $9{:}\ \_ = 3$	$3 + \_ = 8$ $6 + \_ = 8$	$5 - \_ = 1$
Congruent with rational numbers	$6 \times \_ = 11$ $\_ \times 2 = 7$	$8{:}\ \_ = 5$ $14{:}\ \_ = 5$	$\_ + 3 = 4.7$ $2 + \_ = 3.3$	$13 - \_ = 7.5$ $8 - \_ = 3.8$
Incongruent (rational numbers smaller than 1)	$\_ \times 4 = 1$ $3 \times \_ = 2$ $8 \times \_ = 3$ $\_ \times 6 = 4$	$6{:}\ \_ = 14$ $2{:}\ \_ = 5$ $5{:}\ \_ = 8$ $3{:}\ \_ = 7$		
Buffers			$\_ + 5 = 2$ $\_ + 7 = 3$	$5 - \_ = 9$ $5 - \_ = 7$ $3 - \_ = 8$

2.2 Materials

A paper-and-pencil test was administered to the students with 42 questions divided into four parts, with the items in each part being given in randomized order. The first part included 28 questions involving arithmetic operations between one given number and one missing number, as well as the result of an operation (e.g., $2{:}\_ = 5$). The tasks that were used are presented in Table 1. The students were asked to decide whether it is possible for a given relation such as this one to be true, by choosing between two given alternatives: “it is possible” and “it is not possible”.

There were three main categories of tasks depending on the type of missing number in the given operation. First, there were congruent tasks with operations between a given natural number and a missing one, for example $7 \times \_ = 21$. Second, there were congruent tasks that involved rational numbers bigger than 1 as missing numbers, for example $6 \times \_ = 11$. Students’ intuitions about the results of the arithmetic operations were not violated in either case; however, to successfully complete the second type of congruent tasks, students were required to think beyond the set of natural numbers. The tasks in the third set were incongruent, as they involved rational numbers less than 1 as the missing numbers—something that violates students’ intuitions about the results of operations (e.g., for $2{:}\_ = 5$, division results in a larger number).

All four operations were used in the tasks above; however, there were no incongruent tasks involving addition or subtraction, since, as explained above, students had not yet learned about negative numbers. This means that for addition and subtraction tasks, students’ intuition that addition results in bigger numbers and subtraction results in smaller numbers is their expected response, because this is what they know so far. Therefore, for tasks involving negative numbers, the expected response was “it is not possible.” These responses were included as buffers to avoid always having “it is possible” responses.

In the second part of the questionnaire the students were given six tasks that involved inequalities, as presented in Table 2. An operation between two numbers was presented in one of the sides of each of the inequalities, with the operation sign missing, for the students to fill in one of two given alternatives: *multiplication* or *division*. The other side of the inequality was filled with one of the initial numbers. There were two congruent tasks involving natural numbers with the result being in line with students’ intuitions about the operations. The remaining four were incongruent tasks involving one natural number and one rational number smaller than 1 (e.g., $6\_0.2 < 6$), such that the result is counter to students’ intuitions about the given operation.

Table 2 Overview of the missing operation tasks by type (congruent, incongruent)

Congruent	Incongruent
$3\ \_\ 10 > 3$	$6\ \_\ 0.2 < 6$
$5\ \_\ 2 < 5$	$4\ \_\ 0.5 > 4$
	$10\ \_\ \frac{1}{2} < 10$
	$10\ \_\ \frac{3}{4} > 10$

In the third part of the questionnaire students were given three sets of rational numbers with four numbers each and were asked to put the numbers in order from smallest to largest; two of the given sets included fractions and the remaining ones involved decimal numbers. The first set only contained unit fractions (i.e., $\frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \frac{1}{11}$, given in that order). The second set contained the number 1 and three fractions either smaller or bigger than 1 (i.e., $\frac{1}{2}, \frac{3}{2}, 1, \frac{1}{4}$). The set of decimals included four decimal numbers with up to three decimal digits (i.e., 0.12, 1.549, 0.4, 0.387). The students were asked to put the numbers in the correct order from smallest to largest.

The last part of the questionnaire contained five forced-choice questions focusing on students’ understanding of the density of the rational numbers. In three of the questions students were asked how many numbers there were between two given pseudosuccessive numbers. The remaining two questions were about the previous and the subsequent numbers of a given one. The items of the first kind—which were offered in randomized order—differed on the type of interval endpoints, namely two integers (i.e., 0 and 1), two decimals (i.e., 0.005, 0.006), and two fractions (i.e., $\frac{1}{3}$, $\frac{2}{3}$). The answer choices were the same across all items, were presented in mixed order, and were as follows: (a) There is no other number in between, (b) There are more numbers in between and we can name them all one by one, (c) There are so many numbers in between that we can not name them all.

Lastly, there were two questions asking students whether there is a number that comes directly before or after a given number (i.e., after 10, before 6.3). The specific question was "Is there a number that is immediately before 10 (after 6.3) and if so, what is that number?" The answer choices were (a) Yes, there is such a number, it's…; (b) Yes, there is such a number, but I cannot tell exactly what it is; (c) No, there is no such a number, because…

The answer choices reflected the three main levels of understanding numbers beyond natural numbers as discussed in the related research literature (Merenluoto and Lehtinen 2002; Vamvakoussi and Vosniadou 2010), namely: (a) an understanding of rational numbers by applying properties of the natural numbers, such as discreteness; (b) an intermediate understanding in which natural number knowledge is interfering, however in a more sophisticated way compared with the first alternative, characterized by an *advanced discreteness* (Vamvakoussi and Vosniadou 2010); and (c) correct understanding of the density, which was considered as correct even if they did not give an explanation.

2.3 Procedure

The students completed the tests in their classroom during their mathematics course with the presence of their teacher and the researcher. Students were told there was only one correct answer for each question. The questionnaire contained instructions and also some examples for the first part, such as that $3 \times \_ = 7$ is possible. For the first part of the questionnaire, students were told to choose one of the two given alternatives that best represents their opinion. They were also explicitly told that they could think with any kind of number they know.

3 Results

Students' responses in the missing number items were scored on a right/wrong basis; null responses were scored as zero. Overall the questions used in the questionnaire showed high reliability (Cronbach's Alpha = 0.716). Analysis of variance of students' total score in the test showed no main effect either for gender [$F(1, 181) = 0.027$, $p = 0.869$, $\eta_p^2 = 0.000$], school [$F(1, 181) = 0.297$, $p = 0.586$, $\eta_p^2 = 0.002$], or grade [$F(1, 181) = 0.292$, $p = 0.590$, $\eta_p^2 = 0.002$], which indicates that older students did not necessarily do better than younger students.

Table 3 Mean scores for each category of the missing number items

Tasks	N	Minimum	Maximum	Mean	Std. deviation
Congruent with natural numbers	189	0.14	1	0.844	0.188
Congruent with rational numbers	189	0	1	0.434	0.192
Incongruent (rational numbers <1)	189	0	1	0.196	0.211
Buffers	189	0	1	0.096	0.186

3.1 Natural number bias in arithmetic operations

For the first part of the questionnaire the mean scores were calculated for each category of the missing number tasks that appear in Table 1; the results are presented in Table 3. As expected, students' highest performance was on congruent tasks that involved natural numbers as missing numbers and the lowest on the incongruent tasks. The fact that very few mistakes were made on the congruent tasks indicates that students understood the tasks, which were well within their abilities. Paired-samples t-tests showed that, in line with our main hypothesis concerning the dual effect of the natural number bias, students scored significantly higher on the congruent tasks that involved natural numbers as missing numbers than on the congruent tasks that involved rational numbers $t(188) = 28.412$, $p < 0.001$ and they also scored significantly higher on the congruent tasks that involved rational numbers than on the incongruent tasks $t(188) = 14.08$, $p < 0.001$. As expected, the vast majority of students responded negatively on the five buffer tasks, and their score in those tasks was significantly lower even than their score in the incongruent tasks $t(188) = 6.576$, $p < 0.001$, and their responses in those tasks were excluded from the analysis that follows.

In the second part of the questionnaire, where students were asked to choose the operation (i.e., multiplication or division) that could make the inequality true, the results showed again that the students did significantly better on the congruent tasks ($M = 0.767$, $SD = 0.359$) compared with the incongruent tasks ($M = 0.416$, $SD = 0.427$), $t(188) = 8.454$, $p < 0.001$, supporting the hypothesis that

Table 4 Mean scores for each category by operation

Category of item	Operation	N	Mean	Std. deviation
Congruent with natural numbers	Addition	189	0.958	0.157
	Subtraction	189	0.915	0.279
	Multiplication	189	0.942	0.169
	Division	189	0.719	0.393
Congruent with rational numbers	Addition	189	0.809	0.339
	Subtraction	189	0.756	0.377
	Multiplication	189	0.269	0.328
	Division	189	0.304	0.370
Incongruent (rational numbers <1)	Multiplication	189	0.120	0.218
	Division	189	0.272	0.308

there are certain intuitions concerning multiplication and division, namely that multiplication produces bigger numbers and division produces smaller ones.

3.2 The effect of natural number bias in each of the given operations

A subsequent analysis focused on the way students responded in each of the categories of congruent/incongruent tasks for each of the four arithmetic operations between numbers and missing numbers. Therefore mean scores were calculated for the cases that included more than one item in each category; the results are presented in Table 4. Keep in mind that there were no incongruent tasks with addition and subtraction.

Paired-samples *t* test indicated that students did significantly better in the congruent tasks that involved natural numbers as missing numbers than in those with rational numbers; this was the case for addition $t(188) = 5.974$, $p < 0.001$, subtraction $t(188) = 4.97$, $p < 0.001$, multiplication $t(188) = 26.039$, $p < 0.001$, and also division $t(188) = 10.591$, $p < 0.001$. Furthermore, students did better in the congruent tasks that involved missing rational numbers bigger than 1 than in the incongruent tasks with missing rational numbers smaller than 1, which violated students' expectation about the result of each operation; this was the case for multiplication $t(188) = 6.513$, $p < 0.001$, and also division, in which case the difference was not significant $t(188) = 1.603$, $p = 0.111$. Surprisingly, students did better in the tasks that involved division than multiplication; this was the case in the incongruent tasks $t(188) = 6.421$, $p < 0.001$, and also in the congruent tasks with missing rational numbers, but this latter difference was not significant $t(188) = 1.08$, $p = 0.281$. This indicates that students appeared more willing to accept that division could make numbers bigger than to accept that multiplication could make numbers smaller, in contrast to the results of Van Hoof and colleagues' (2014) study with older students (eighth to twelfth graders).

Table 5 Performance in ordering of rational numbers tasks

Question	Wrong	Right
Ordering of unit fractions	106 56.1 %	83 43.9 %
Ordering of decimals	120 63.5 %	69 36.5 %
Ordering of fractions	140 74.1 %	49 25.9 %

From the results it can be also concluded that students had a mean accuracy of 75–81 % on congruent addition and subtraction tasks that involved decimal numbers as missing numbers (e.g., $\_ + 3 = 4.7$), while their accuracy on congruent multiplication and division tasks that involved decimal numbers as missing numbers (e.g., $6 \times \_ = 11$) was 26–31 %. A possible explanation for this difference is that it is much easier to find the missing rational number in the case of addition or subtraction than in the case of multiplication or division. Despite the fact that students were specifically told that they did not have to find the specific number that is missing, upon easily finding the missing number it is probably also easier to respond that such a number exists and thus that the specific result is possible. However, more research is needed to further clarify this issue.

3.3 Natural number bias in understanding ordering and density of rational numbers

Responses on the three tasks that required students to order two sets of fractions and one set of decimal numbers were coded as right/wrong; the results are presented in Table 5. As can be seen, fewer than half of the students succeeded in each of these tasks. The influence of the natural number bias on students' ways of ordering given numbers is more obvious when focusing on the kinds of mistakes the students made. In the case of ordering the unit fractions, almost all of the students who responded incorrectly (94 students, 49.7 %) ordered the fractions by applying the rule "the bigger the denominator the bigger the fraction," which is in line with an understanding of rational numbers affected by the properties of natural numbers. Also in line with this way of understanding rational numbers were students' mistakes on the ordering of decimal numbers, in which 68 students (36 %) ordered the numbers by following the rule "the longer the number, the larger." Lastly, in the case of ordering three fractions compared to 1, two main categories of incorrect responses appeared: an ordering in which all the fractions were bigger than 1 (55, 29.1 %), and the exact opposite, an ordering in which all the fractions were smaller than 1 (49, 25.9 %).

Performance on the last set of questions that included five tasks that tested students' understanding of the density of rational numbers is presented in Table 6. Fewer than one in three students responded correctly to the questions about how many numbers there are between two pseudosuccessive numbers, and very few students responded correctly to the tasks about the numbers that come directly before and after a given number. Among the first three tasks, the largest percentage of intermediate responses corresponded to how many numbers there are between 0 and 1, and students' worst performance was on how many numbers there were between 0.005 and 0.006. This shows that when asked about the density of an interval of numbers, students tended to assign different behaviors to natural numbers, decimals, and fractions. This finding is consistent with prior research in the field (Vamvakoussi and Vosniadou 2010). Students also appeared to face strong difficulties in the tasks concerning the immediate next and previous number of a given one; less than 2 % of the students succeeded, and the majority of incorrect responses indicated 9 as the number directly before 10. These results were probably due to the counter-intuitiveness of the questions and the fact that it was not specified in the questions that the reference was not exclusively to natural numbers, even if students were told on the general instructions that they could think with any number they knew. Therefore those two questions were excluded from the rest of the analysis.

Table 6 Performance in each of the items about the density of rational numbers

	Wrong	Intermediate	Right
Numbers between 0 and 1	63 33.3 %	69 36.5 %	57 30.2 %
Numbers between 0.005 and 0.006	112 59.3 %	39 20.6 %	38 20.1 %
Numbers between 1/3 and 2/3	95 50.3 %	55 29.1 %	39 20.6 %
Directly before 10	170 89.9 %	16 8.5 %	3 1.6 %
Directly after 6.3	167 88.4 %	19 10.1 %	3 1.6 %

3.4 Correlations between the different aspects of the natural number bias

In order to create a variable that could best describe the natural number bias in arithmetic operations with missing numbers, students' score on the first part of the questionnaire included only their responses on the tasks, either congruent or incongruent, that involved rational numbers as missing numbers, and not the congruent tasks that involved natural numbers. Table 7 presents the correlations between students' performance on the operations including missing number tasks, with their performance on the missing operation tasks, and with their performance on rational numbers ordering task. Table 8 presents the correlations between performance in the missing numbers tasks and the tasks about understanding the density of the rational number set. As shown in Table 7, Pearson test of correlation indicated that there was a significant positive correlation between students' scores on the arithmetic operation tasks including missing numbers and students' scores on the missing operations tasks. Also, students' performances on arithmetic operations with missing numbers are positively correlated to their ability to order decimals and fractions. Students' abilities to order fractions and decimal numbers are highly correlated as well. On the other hand, students' performances on the missing operation tasks were not significantly correlated to their ability to order any of the given sets of rational numbers.

As shown in Table 8, students' performance on the arithmetic operations with missing number tasks was positively correlated with their understanding of the density of rational numbers but not between understanding of the density of rational numbers and students' performance in the missing operation tasks. Out of the results presented in Tables 7 and 8 the positive correlation between students' responses in the operations with missing numbers tasks and their ability to order rational numbers and reason about the density of their structure may be driven by their common characteristics, namely, that they may entail reasoning by mainly using number knowledge. On the other hand, students may respond in the missing operation tasks using only their intuitive knowledge and general rules about the

Table 7 Correlations between performance in the ordering tasks and the operation tasks

	Score on operations with missing numbers	Score on the missing operation items	Ordering of unit fractions	Ordering of decimals
Score on the missing operation items	0.172*			
Ordering of unit fractions	0.146*	0.082		
Ordering of decimals	0.163*	0.131	0.325**	
Ordering of fractions	0.219**	0.069	0.523**	0.354**

* $p < 0.05$, ** $p < 0.001$

Table 8 Correlations between performance in the operation tasks and the density of rational numbers tasks

	Score on operations with missing numbers	Score on the missing operation items	Numbers between 0 and 1	Numbers between 0.005 and 0.006
Score on the missing operation items	0.172*			
Numbers between 0 and 1	0.139*	−0.019		
Numbers between 0.005 and 0.006	0.144*	0.070	0.418**	
Numbers between 1/3 and 2/3	0.148*	0.010	0.349**	0.462**

* $p < 0.05$, ** $p < 0.001$

results of operations without caring about the numbers involved, and this is reflected in weak correlations between their responses on those tasks.

4 Discussion

The present study further investigates the influence of the natural number bias on students' ways of reasoning about arithmetic operations providing quantitative data about the way this bias acts in this learning domain. In addition, it investigates the way the natural number bias correlates with certain aspects of students' understanding of the number concept. In a paper-and-pencil test, fifth and sixth grade students were given congruent and incongruent tasks, which involved operations between given numbers or between given numbers and missing numbers, as well as questions about the order and density of the rational numbers.

The results supported our main hypothesis that in the domain of operations between numbers there is a natural number bias that interferes with students' reasoning, which acts in two main ways. First, it supports, and probably also shapes, students' tendencies to intuitively associate each operation with specific results, that is, that the result of multiplication is always bigger than the numbers involved and the result of division is always smaller. Second, in operations between numbers that are not presented with specific numerals, the natural number bias causes students to mentally substitute only with natural numbers, deciding on a general result based on the results of such trials.

The former conclusion is supported by the finding that the students in our sample performed significantly better in the congruent tasks that were designed in line with their intuitions (i.e., multiplication produces bigger numbers), compared with the incongruent tasks that violated those intuitions. Not only in the missing number tasks, but also in the second part of the questionnaire in which students were asked to choose the proper operation to make each of the given inequalities true, students showed a strong tendency to rely on their intuitions about the results of multiplication and division. This is in line with previous findings that school children, as well as adults, hold strong intuitions when considering the results of operations, appearing in operations between numbers and algebraic expressions (Greer 1989; Vamvakoussi et al. 2012; Van Hoof et al. 2015), or in word problems (Bell et al. 1981; Fischbein et al. 1985; Harel and Confrey 1994).

The latter conclusion is supported by the significant differences between the results of two kinds of congruent tasks that were administered to the students in the missing number tasks—those that involved natural numbers and those that involved rational numbers as missing numbers—even though in both cases students' intuitions about the effects of the operations were not violated. The congruent tasks that involved rational numbers as missing numbers (e.g., $6 \times \_ = 11$), elicited significantly more incorrect responses than the congruent tasks that involved natural numbers as missing numbers (e.g., $7 \times \_ = 21$); something that was noted in all four operations. This innovative finding shows that students do not necessarily decide on the result of an operation using their preconceptions about the effect of the arithmetic operations, that is, by focusing exclusively on the operation sign without considering the numbers involved; instead, students may decide on the result of an operation after testing (mentally) some random numbers. The natural number bias also affects this way of reasoning by biasing the kinds of numbers that students tend to substitute the missing numbers with, namely natural numbers.

Previous studies had provided some indications in support for this claim. Responses in individual interviews with students from eighth to twelfth grade showed that they tended to use the above two mentioned strategies when reasoning about the effect of arithmetic operations (Christou and Vosniadou 2012; Van Hoof et al. 2014). In addition, research on students' interpretations of the kinds of numbers represented by literal symbols in algebraic expressions showed that students give priority to natural numbers (Christou and Vosniadou 2005, 2012). The innovative character of the results presented in this study is that they support this claim by providing quantitative data and statistical analysis of data coming from a large group of participants and their responses in a series of mathematical tasks. Those tasks were designed to be either in line with or counter to students' intuitions about the results of operations, and also to differentiate concerning the kinds of numbers involved. In this way they managed to capture and reveal the dual aspect of the effect of the natural number bias in the mathematics-learning domain (see also Christou, submitted).

Another important finding of this study is that these two ways of reasoning about the results of arithmetic operations, namely the use of generalized rules or the strategy to try with specific numbers, are related to each other in quite complicated ways. Students' performance on the tasks that involved operations between missing numbers, which entail both strategies, appeared to be positively correlated to their performance on the tasks in which they were asked to choose a correct operation that would make an inequality true. In addition, there was a strong correlation between reasoning about arithmetic operations with missing numbers and other aspects of number reasoning such as ordering rational numbers and arguing about their dense structure. On the other hand, students' performance on these tasks about number reasoning was not correlated to their performance on the missing operations tasks. A possible explanation for this is that responding to the missing number tasks entails focusing on the numbers involved, which is also the case when reasoning about ordering or about the density of the rational number set. This is, however, not the case when responding to the missing operation tasks, which could be done by referring to the general rules about the effect of each operation without caring about the numbers involved, and this could explain why responses in these tasks were not correlated to responses on the tasks about ordering and density of rational numbers, which both mainly entail reasoning about numbers. In other words, out of these results it could be argued that when the students are dealing with operations between numbers it is one thing what *numbers can do* (i.e., how they can be ordered or how dense their structure is), and another what *operations can do* (e.g., multiplication makes numbers bigger). Consequently, these two aspects of reasoning in arithmetic operations could be quite distinctive from each other; however, each of them may be influenced by the natural number bias, as explained above. Additional research is needed to further elaborate on the way these two strategies on reasoning about arithmetic operations actually relate to each other.

4.1 Inhibiting the natural number bias

The natural number bias is neither due to poor understanding of a certain concept nor due to failing to apply a concept or its properties in certain conditions. It is rather the result of inappropriate application of correct natural number rules in cases that do not apply, and the tendency to assign only natural numbers in cases of missing number tasks where any real number would apply. This means that in situations entailing reasoning with non-natural numbers, the interference of natural number knowledge must be inhibited.

The term *inhibition* has been used throughout the history of research in psychology and cognition in different ways and in many different contexts (for a thorough discussion see MacLeod et al. 2003). Here, this term is used to characterize the cognitive process of deliberately inhibiting automatic or prepotent responses, produced by innate activation of dominant schemas, such as the counting number schema. It is argued that for students to be able to reason with numbers other than natural numbers and to overcome their intuitions about the effects of the arithmetic operations (i.e., to accept that under certain conditions multiplication may also produce smaller numbers), they would need to inhibit the natural number knowledge interference, meaning their automatic responses based on knowledge of natural numbers.

To do so, students would need to develop certain kinds of inhibition strategies. The need for inhibition strategies appears frequently in studies of intuitions on arithmetic operations, taking different names such as *stop and think* strategies (Vamvakoussi et al. 2013), *alarm devices* (Fischbein 1990) or *critics* (Davis 1984). One inhibition strategy, for example, is to always try with at least one non-natural number—a negative or a number smaller than 1—in cases of missing number tasks, or to have in mind that "multiplication does *not always* make bigger."

Developing and implementing such inhibitory strategies presupposes that students are aware of their intuitions about the properties of numbers. As noted before, students not only carry intuitive ideas about numbers and the way that these numbers act on operations, but they might not be conscious of carrying them (Vamvakoussi and Vosniadou 2010). Certain examples and counterexamples can be used in provoking and also refuting certain kinds of erroneous intuitions students may hold. Another presupposition is that students have developed a deeper conception of number beyond the natural numbers that is closer to the mathematical concept of number, such as incorporating the rational and the real numbers (Smith et al. 2005). Developing a more sophisticated conception of number involves learning with revision of the prior knowledge, which is more time-consuming and does not develop automatically, requiring substantial cognitive effort (Vamvakoussi and Vosniadou 2010). In line with this view, the results of the current study showed that there were no significant grade differences on the multiplication and division tasks, indicating that older students do not necessarily do better than younger students. It is not by chance that even educated adults and also pre-service and elementary teachers still carry the same intuitions about the results of operations (Graeber et al. 1989; Vamvakoussi et al. 2012).

However, there seems to be room for cultivating such inhibitory strategies through specifically designed learning interventions that focus on specific erroneous intuitions. Certain interventions, such as triggering an intuitive incorrect response and then falsifying it, could be used in constructive ways for raising students' awareness of their intuitions and for developing certain strategies to remedy them (Christou 2012; Babai et al. 2015). To this end, the *refutational argumentation* methodology (Hynd 2001) could be fruitfully used as a means not only to challenge students' erroneous beliefs but also to offer them alternative conceptions to adopt. This argumentation uses the cognitive conflict strategy by means of provoking and also falsifying students' anticipations about the results of arithmetic operations that stem from their existing beliefs, in a more constructive way, with promising preliminary results (Christou 2012). Some of these suggestions could be adapted and applied in interventions that would target the natural number bias in operations between numbers.

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