Chapter 4

Observing Mathematical Problem Solving through Task-Based Interviews

Gerald A. Goldin

Over a period of 2 decades, mathematics education has evolved to stress conceptual understanding, higher-level problem-solving processes, and children's internal constructions of mathematical meanings in place of, or in addition to, procedural and algorithmic learning (Davis, Maher, & Noddings, 1990; von Glasersfeld, 1991). With this trend, the structured clinical interview has found greater acceptance as a research method. It lends itself well to the qualitative study and description of mathematical learning and problem solving without the exclusive reliance on counts of correct answers associated with pencil-and-paper tests.

In general, such structured interviews are used in research for the twin purposes of (a) observing the mathematical behavior of children or adults, usually in an exploratory problem-solving context, and (b) drawing inferences from the observations to allow something to be said about the problem solver's possible meanings, knowledge structures, cognitive processes, affect, or changes in these in the course of the interview.

For me, structured interviews are especially attractive as a means of joining research with educational practice. Reforms in school mathematics in the United States endeavor (among other goals) to foster the discovery of patterns and ways of reasoning about them and to develop skill in constructing original, nonstandard solution methods. Guided explorations by children and small-group problem solving are encouraged. These goals supplement (if they do not actually supplant) more "traditional" teacher-centered, direct instruction emphasizing mastery of standardized mathematical representations, rules, and procedures. In the

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reformed context it becomes increasingly important to be able to describe and assess the longitudinal mathematical development of individual children. We need to find ways of observing that permit valid inferences about the deeper understandings that the new emphases try to develop (Lesh & Lamon, 1992). Thus, task-based interviews have importance both as research instruments and as potential research-based tools for assessment and evaluation. They offer the possibility of obtaining information from students that bears directly on classroom goals and can help answer research questions central to the educational reform process: What long-term consequences are innovative teaching methods having for children's mathematical development? What powerful problem-solving processes (if any) are students learning in "reform" classrooms? What cognitive representational structures are they developing? Are all children developing these, or only some? What are the affective consequences of reform? What beliefs about mathematics are children acquiring?

The main purpose of the chapter is to discuss some of the scientific underpinnings of task-based interview methodology in the study of mathematical problem solving. I touch on a set of issues having to do with the reproducibility, comparability, and generalizability of research findings. The importance of having an explicit theoretical perspective when structuring an interview is discussed, as well as the fact that choices made during interview design can result in foreseeable consequences—for instance, obtaining some information at the expense of other information. I try to be aware throughout the chapter of constraints and limitations imposed by the social and psychological contexts of interviews as well as of the interplay among task variables, contextual factors, observed behaviors, and cognitions inferred by the researcher.

The main points are illustrated with reference to five structured, individual interviews, designed around mathematical problem-solving tasks for the purposes of a longitudinal study. These provide concrete examples related to the central questions. The views described here helped shape the development of the scripts for these interviews and were in their turn considerably influenced by that process. What we learned in developing the interview scripts, carrying out the interviews, and interpreting the results influenced some principles of interview design and construction that are suggested for consideration by the mathematics education research community.

QUESTIONS RAISED BY TASK-BASED INTERVIEW RESEARCH

Whether we regard task-based interviews as research instruments or as assessment tools, their use to observe and draw inferences from mathematical behavior raises fundamental questions. It is my view that future research studies involving clinical interviews would benefit greatly by giving explicit, advance consideration to the following questions:

1. In what sense do the interviews permit genuinely scientific investigations? By this I mean to inquire about the implications of the task-based interview

methodology employed for (a) the examination, analysis, and communication to others of the measurement process, (b) the replicability of results (from one interview to another with the same subject, from one population to another with similar characteristics, from the current study to other studies, and so forth), (c) the comparability of outcomes across studies that may employ different interview instruments, and most important, (d) the eventual *generalizability of the findings* that are obtained from the observations made.

2. What role does theory play in structuring the interviews? To what extent are the observations made during an interview contingent on the tacit or explicit theoretical assumptions that underlie the interview questions and procedures? How does theory guide the choice of questions in the interview? How does it guide the contingencies that are planned for? How does it allow for unplanned contingencies? How are we to draw inferences about cognition, affect, or both, from our observations? What is the interplay among task variables (the characteristics of the problems on which task-based interviews are based), observed behaviors, and the inferences we can draw? How should we come to modify, substantially revise, or even discard our theories on the basis of the empirical outcomes of the interviews?

3. What constraints or limitations are imposed by the social, cultural, and psychological contexts of the interviews? How may the student's expectations, presumptions, apprehensions, and intentions interact with mathematical cognitions and affect (and with task variables) to influence the interview outcomes?

The intent in raising these questions is to begin the discussion from a scientific perspective, offering illustrative examples from the current study, and to propose some preliminary and partial answers—in the context of that study—that may be more generally applicable. My goal is to frame some general principles of interview design and construction that may be appropriate for the mathematics education research community to adopt. For example, it may be possible to characterize the trade-offs that take place as questions are selected for incorporation in an interview script and, through explicit principles, to optimize the information gathered in a task-based interview.

The ideas advanced here have their origins in earlier studies of mathematical problem solving and in discussions about observation, measurement, and assessment (Bodner & Goldin, 1991a, 1991b; Cobb, 1986; DeBellis & Goldin, 1991; Goldin, 1982, 1985, 1986, 1992a; Goldin & Landis, 1985, 1986; Goldin & McClintock, 1980; Hart, 1986). But they are immediately instigated by a series of task-based interviews that a group of us at Rutgers University created in the context of a longitudinal study of individual elementary school children's mathematical development. Five scripts were written, and used from 1992 to 1994, as the basis for a series of individual problem-solving interviews with children. The next section describes these briefly. I then return to explore aspects of the scientific nature of task-based interviews and address the role of theory and the role of context. The chapter concludes with comments concerning principles of interview design and construction.

AN EXPLORATORY LONGITUDINAL STUDY

In a study whose outcomes are still being analyzed, the mathematical development of an initial group of 22 children was observed for approximately 3 years. At the outset, in the 1991–92 school year, subjects were 8 to 10 years old. They were then in the third and fourth grades in a cross-section of New Jersey's public schools: two urban schools (5 third graders and 4 fourth graders); one school in a predominantly blue-collar, "working class" community (7 fourth graders); and one school in a suburban, "upper middle class" district (6 third graders). These schools, and the children's teachers, were participating in an intensive, constructivist-oriented mathematics teacher development–mathematics education reform partnership called MaPS (Mathematics Projects in Schools), sponsored by the Rutgers Center for Mathematics, Science, and Computer Education and the Graduate School of Education and directed by Carolyn A. Maher and Robert B. Davis. In fact, one reason for initiating the longitudinal study—for which data sources included videotapes of the children's individual problem solving, as well as their small-group mathematical activity inside and outside class—was to be able to assess some of the project's outcomes in relation to individual children's mathematical understandings as they grew over time.

One component of this study consisted of a series of task-based, individual interviews with each child over a part of the 3 years, conducted under the direction of the author (DeBellis & Goldin, 1993; Goldin, 1993; Goldin, DeBellis, DeWindt-King, Passantino, & Zang, 1993). Five interviews were designed and administered between spring 1992 and Spring 1994, with the goals of observing complex, individual mathematical problem-solving behavior in detail and drawing inferences from the observations about the children's thinking and development. Thus, this component of the study was, from a scientific standpoint, mainly exploratory and descriptive—subjects were not a random sample from a larger population, and no general hypotheses were being explicitly tested. Rather, we hoped to describe individual mathematical development in as much detail as possible, focusing not on standard, discrete skills or algorithmic problem solving, but on the growth of complex, internal representational capabilities. Tied to these goals, the interview design included several steps: (a) planning in relation to mathematical content and structure, anticipated observations, and inferences—discussed further in the next two sections; (b) creating an interview script, and its critique by the research group in a graduate seminar; (c) pilot testing the script in a different school, with children not part of the longitudinal study, and revising it on the basis of the pilot test; and (d) training and rehearsing with clinicians, including practice sessions. Initially we hoped that half or more of the 22 children would remain in the study for the full term; originally six interviews were planned, but funding constraints limited us to five. As it turned out, 19 of the original group of children participated in all five interviews.

The interviews themselves were designed to take less than one class period. In every interview, alternative embodiments for external representation were given to the child: paper and pencil, markers, cards, chips or other manipulatives, paper cutouts, a hand calculator, and so on, in accordance with the task. The questions within an interview tended to increase in difficulty, so that each child began with a level of comfort, but even mathematically advanced children encountered some questions that were challenging before the interview ended. Free problem solving was encouraged wherever possible, with (specified) hints given or suggestions made only after the child had the opportunity to respond spontaneously. All responses were accepted by the clinician (with occasional exceptions, specified in advance); thus "wrong" and "correct" answers were treated similarly. Follow-up questions by the clinician were asked without an overt indication of the correctness of earlier responses. Two video cameras operated simultaneously during each interview—one focusing on the clinician and the child or the child's face, the second focusing on the work the student was doing (working with paper and pencil or handling manipulatives); in Interview 3, a third camera also provided a close-up of the child's facial expressions. An observer made notes during the interview. Subsequently the videotapes were transcribed, viewed, and analyzed. What follows is a capsule description of each interview script. The full texts of the interviews are available for research purposes from the Rutgers Center.

Task-Based Interview 1

The first interview script (55 pages, about 45 minutes) was written during 1991–92 and administered in May and June 1992. The task, based on a high school–level problem of the National Assessment of Educational Progress, involves laying out for the child three cards, one at a time (see Figure 4.1): "Here is the first card, here is the second card, and here is the third card."

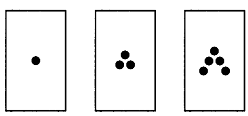

Figure 4.1. The first three cards presented in Task-Based Interview 1.

The cards are drawn from a stack in an envelope, so the child may infer from the context that there is a deck larger than the few cards shown and (possibly, tacitly) may also infer that there is a pattern present. After a brief pause to allow a spontaneous response, the child is asked,

- "What do you think would be on the next card?"

The materials placed ahead of time on the table are blank index cards (the same size as those with dots), felt-tipped markers of different colors, round red and black chips (checkers), a pad of paper, and a pencil. The child can use any items.

A series of exploratory questions follows, with contingencies based on the nature of the responses and special emphasis on exploring the child's pattern construction and use of external representations. After a complete, coherent response to the first question has been elicited, the child is similarly asked the following questions in slow succession:

- "What card do you think would follow that one?"
- "Do you think this pattern keeps going?"
- "How would you figure out what the 10th card would look like?"
- "Here's a card [showing 17 dots in the chevron, or inverted V, pattern]. Can you make the card that comes *before* it?"
- "How many dots would be on the 50th card?"

The script is written so that for each main question, exploration proceeds in four stages: (a) posing the question ("free" problem solving) with sufficient time for the child to respond and only nondirective follow-up questions (e.g., "Can you tell me more about that?); (b) heuristic suggestions if the response is not spontaneous (e.g., "Can you show me by using some of these materials?"); (c) guided use of heuristic suggestions, again to the extent that the requested description or behavior does not occur spontaneously (e.g., "Do you see a pattern in the cards?"); and (d) exploratory (metacognitive) questions (e.g., "Do you think you could explain how you thought about the problem?"). The clinician's goal is always to elicit (a) a complete, coherent verbal reason for the child's response and (b) a coherent external representation constructed by the child, before going to the next question (for the question about the 50th card, an external representation is not required). A complete, coherent reason means one based on a described or modeled pattern, but this pattern is *not* required to be "canonical" (i.e., to have the 4th card drawn with 7 dots in the chevron pattern) for the response or external representation to be considered complete and coherent.

This "nonroutine" task embodies an additive structure in an arithmetic sequence represented through a geometric arrangement of dots. It provides opportunities for the child to detect numerical or visual patterns, or both; to use visual, manipulative, and symbolic representations; and to demonstrate reversibility of thinking.

Task-Based Interview 2

The design for the second interview script (38 pages, up to about 55 minutes) was completed in fall 1992. The script was used in individual interviews administered during winter 1993 with the same children (then in fourth and fifth grades). As in the first interview, materials (a pad, a pencil, markers, and checkers) are placed ahead of time on the table in front of the child. First some preliminary questions are asked with the intent of exploring the child's imaginative and visual processes: The child describes whether she or he is right- or left-hand-ed. Then the child is asked to imagine a pumpkin, to describe it, to manipulate the image in various ways (including cutting the pumpkin in half), to spell the word *pumpkin,* to spell it backward, and to talk about these activities. A series of mathematical questions follows. For each, the follow-up includes (where appropriate): "Can you help me understand that better?" or "Are there any other ways to take (one half) (one third)?" or both questions.

- "When you think of one half, what comes to mind?"
- "When you think of one third, what comes to mind?"
- "Suppose you had 12 apples. How would you take (one half) (one third)?"
- [Next cutouts are presented in succession: a square, a circle, and a 6-petal flower. For each, the child is asked] "How would you take (one half) (one third)?"
- [Circle cutouts are presented to the child, first with (one half) (one third) (one sixth) represented conventionally (as in a pie graph), then with the same fractions represented unconventionally (the part representing the fraction at the center of the circle). In each case the child is asked] "Can this card be understood to represent (one half) (one third)? (Why?) (Why not?)"
- [A 3-by-4 array consisting of 12 circles and 6-petal flowers is now presented.] "How would you take (one half) (one third)?"
- The child is also asked to write and interpret the usual notation for the fractions one half and one third.

Next a solid wooden cube is shown. Some preliminary questions are asked about its characteristics (number of faces, edges, and corners). The child, guided as necessary toward understanding what these mean, is then asked to think about cutting the cube in various ways:

- "Now think about cutting this cube in half. What would the two halves look like?"
- "Suppose we painted the cube red and then cut it the same way. How many faces are painted red, for the smaller pieces you told me about?"

Similar questions follow about cutting a series of up to five additional cubes, depending on the time available. These cubes are marked with lines at designated vertical or horizontal positions, or both, which results in mutually congruent pieces that are respectively $^1/_3$, $^1/_4$, $^1/_8$, $^1/_9$, and $^1/_{27}$ the volume of the original cube. The script contains numerous suggested exploratory questions and a series of retrospective questions at two different points. This interview thus provides opportunities for the children to express a variety of conceptual understandings related to one half and one third, in many different embodiments in both two- and three-space dimensions. A multiplicative structure is embodied in cutting the solid wooden cube across different dimensions, and special emphasis is placed on exploring visualization by the child.

Task-Based Interview #3

The third interview script (28 pages, about 50 minutes) was completed in May 1993 and administered during May and June of that year. It begins with some introductory questions designed to elicit some of the child's affect in relation to mathematical problem solving: "Could you think back to the first time you remember doing mathematics? What do you remember?" "What is the earliest you remember doing math in school?" "Did your (parents) (brothers or sisters) ever do mathematics with you? Did they like to do mathematics?" "Do you remember doing puzzles or playing games at home? What games did you play?" "Did you ever see or do mathematics on TV?" "Do you remember doing mathematics with friends?" Each question is followed up at the clinician's option; for example, "When did that happen? How old were you then? Could you tell me more about what happened?" In all instances, the child is asked, "How did you feel about that?" or "How did you feel when that happened?" and, if not yet described, "Did you enjoy it? Was there anything you didn't like about it? How do you feel about mathematics now? Do you think this ... has anything to do with how you feel about mathematics now?"

The child is also asked, "Do you think you are good at solving problems?" "What do you think makes someone a good problem solver?" "Who do you think solves problems best in your class? Why do you think (name) is a good problem solver?"

Two different sets of problems are then presented successively: (a) cutting a birthday cake (without or with frosting) to share equally among two or three children and (b) moving colored jelly beans back and forth between two jars. Both problems embody symmetry and coordination of conditions—the first in the context of volume and area, the second in a numerical context. Emphasis is on exploring the child's affect as well as his or her metacognitions about the two tasks. Materials on the table are a ruler; markers; pencils; a pad of blank paper; scissors; sheets of graph paper; a spool of string and a length of cut string; construction paper, Styrofoam shapes with rectangular, circular, and triangular bases; and jelly beans.

The main birthday cake questions were the following:

- "Which would be easier, to cut a birthday cake into three equal pieces or four equal pieces? Why? Could you explain that to me?"
- "Does the shape of the cake matter?"
- "Suppose the cake has icing on the top and on the sides. (Four) (three) people are at the birthday party. How would you cut the cake so that each person gets an equal amount of cake and an equal amount of icing?"

After various explorations, brought to a close when 25 minutes have passed since the start of the interview, the child is encouraged to retrospect with additional questions. Then two transparent glass baby-food jars with twist-off lids, each filled nearly to the top with jelly beans, are presented to the child. One has

100 orange jelly beans and is labeled “ORANGE”; the other has 100 green jelly beans, and is labeled “GREEN.”

- “The next problem is about jelly beans. This jar has 100 green jelly beans [points to the green jar], and this jar has 100 orange jelly beans” [points]. Suppose you take 10 green jelly beans from the green jar and put them into the orange jar [points] and mix them up [pretends to transfer the jelly beans, but does not do it]. Then suppose you take 10 jelly beans from this mixture and put them back into the green jar [pretends]. Which jar would have more of the other color jelly beans in it? Would there be more green jelly beans in the orange jar, or would there be more orange jelly beans in the green jar?”

If the child does not do so spontaneously, he or she is first encouraged to try the experiment and, if necessary, is guided to do so as follows:

- “Could you show me how to do it with the jelly beans? Let’s try the experiment ...”
- “Will it always come out that way? Why do you think so?”

After the student has expressed a firm conclusion, the clinician asks follow-up questions and a final set of retrospective questions focusing on affect as well as on cognition.

Task-Based Interview 4

Interviews 4 and 5 return to selected mathematical ideas from the first two interviews. Interview 4 (41 pages, up to about 55 minutes) again explores the child’s strategic and heuristic thinking in the context of sequences of cards, in close parallel with Interview 1. Materials this time include a hand calculator. Four problems, depicted in Figures 4.2a–d, are presented in succession, in the format of Interview 1: “Here is the first card, here is the second card, and here is the third card.” After a brief pause to allow a spontaneous response or detection of a pattern in Problem 1, the child is asked, “What do you think would be on the next card?” and questions are posed as in Interview 1. After several questions, or after 15 minutes, Problem 2 is presented (see Figure 4.2b), without the clinician removing the cards of Problem 1 that were discussed. After further exploratory questions, Problem 3 is posed (Figure 4.2c), and after additional questions the child is given Problem 4 (Figure 4.2d).

The key follow-up questions in all four problems are similar to those in Interview 1. Once the child has given both an external representation (for Problem 1 only, a good verbal description is accepted) and a coherent reason, the clinician moves to the next problem. As usual, suggestions are made only when the child reaches an impasse. If the child does not spontaneously detect relationships between problems, the clinician asks about this after Problem 2. During the final retrospective, the first 3 cards of Interview 1 (Figure 4.1), with which the children engaged a year and a half earlier, are laid out. Gesturing to

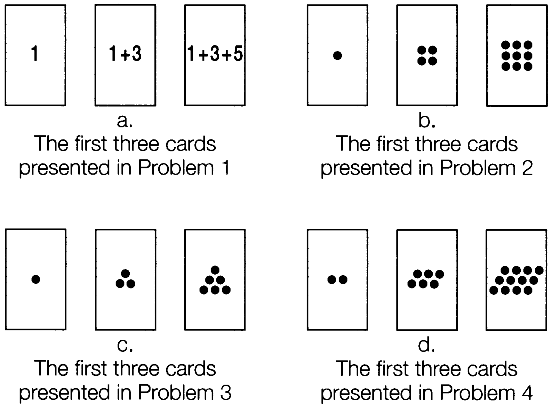

Figure 4.2. The sequence of tasks in Task-Based Interview 4.

all the cards, the clinician asks if the child sees any way to relate today's cards to the previous cards.

Task-Based Interview 5

Interview 5 (27 pages, up to about 55 minutes) also returns to selected mathematical ideas from the earlier interviews, particularly fractions related to $^{1}/_{2}$, $^{1}/_{3}$, and 1/4 as explored in Interviews 2 and 3. Materials given to the student include scissors; a 12-inch ruler marked in both inches and centimeters; an 18-inch length of white curling ribbon; paper circles, squares, and triangles; a pile of red and white plastic chips; a calculator; paper and pencils; and a solid piece of wood the approximate shape and size of a stick of butter, measuring 1" × 1" × 5". The interview begins with open-ended questions about fractions: "When you think of a fraction, what comes to mind?" "Can you tell me more about that?" "Can you show me what you mean?" "Have you studied fractions in school yet?" "What (else) have you studied about them?" "Do you like fractions?" "What do (don't) you like about them?"

The child is then given a sheet of pink paper with five fractions written on it and is asked a series of questions; as always, spontaneous problem solving is allowed before the next question:

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{6}$$

- "What fractions do you see here?" "Can you explain ...what one of these fractions means?" "Why is it written this way?" "Could you show me [using] the materials?"
- "Which fraction is the (smallest) (largest) fraction in the group?" "Why?" "Could you show me what you mean?" "Are there any fractions in this group

that are the same size?" "(Why?) (Why not?)" "Could you show me what you mean?"

Next some pictorial representations on a sheet of yellow paper and new questions are given (see Figure 4.3):

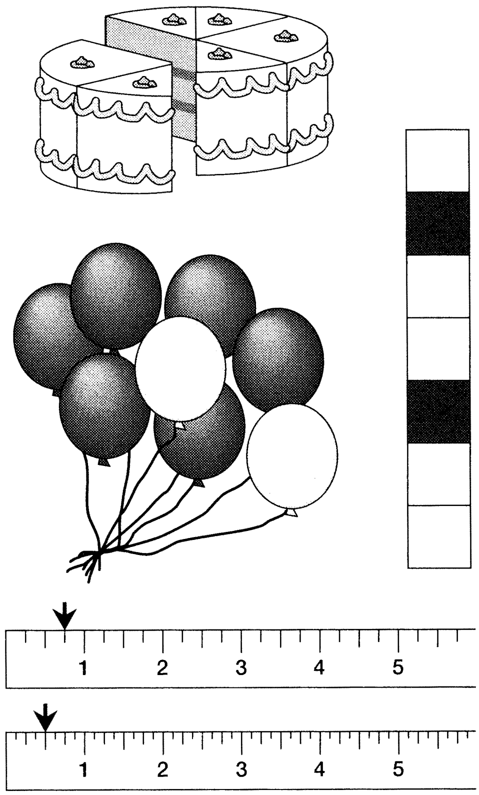

Figure 4.3. Pictorial representations presented during Task-Based Interview 5.

- "Could you use a fraction to describe any of these pictures?" "What fraction or fractions would you use?" "Why?" "Could you show me what you mean?"

All spontaneous answers are accepted, after which the clinician asks about the pictures the child may have omitted and whether the pictures on the yellow sheet of paper go with each other or with the fractions on the pink sheet.

The child is next given a sheet of blue paper with five new fractions written on it.

$$\frac{5}{5} \quad \frac{3}{1} \quad \frac{5}{4} \quad \frac{11}{8} \quad \frac{10}{8}$$

For the balance of the interview, the child solves up to four problem tasks, one at a time), each accompanied by exploratory, nondirective questions. It is not expected that all problems will be completed. When 5 minutes remain, the clinician skips to the final retrospective:

- [A circular shape is presented.] "How could you show one third of this shape?" "Why is that one third?" "Is there any other way to show one third?" "How could you show one fourth of this shape?" "Why is that one fourth?" "Is there any other way to show one fourth?"
- [The 1" × 1" × 5" piece of wood is presented.] "Pretend this is a stick of butter. You need a tablespoon of butter to make a cake. You don't have a measuring spoon, but you know that there are 8 tablespoons in a stick of butter. Here is the butter. How could you find exactly one tablespoon?" [If the answer is imprecise, ask once] "Is there any way to find out more exactly?"
- "Imagine a big birthday cake shaped like a rectangle. Can you imagine what it looks like?" "Describe what it looks like." "Now imagine that there are 12 people coming to the birthday party and they each want a piece of cake. Your job is to cut the cake so that each person gets the same-size piece. How will you cut the cake?" "Could you show me what you mean?" "Are there any other ways to cut it?" [The clinician continues to explore cutting the cake, including the situation of icing on the cake.]
- "A toymaker found some wooden shapes in the corner of her workshop. Some were squares, and some were triangles. She decided to put them together to make little houses [demonstrates using a square and a triangle). The squares looked like this [gestures to the pile of squares]. The triangles looked like this [(gestures to the pile of triangles]. The houses looked like this [places the triangle on top of the square to make a figure that looks like that shown in Figure 4.4]. After a while, she noticed that she had matched exactly $^3/_5$ of the squares with exactly $^2/_3$ of the triangles. How many squares and triangles were there to start with?" "Using these materials, could you show me how she did that?" [If time permits:] "Could there be a different number that works?" ...

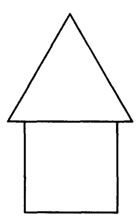

Figure 4.4. House composed of a square and a triangle.

After each of these four problems, the child is asked, "Have you ever done a problem like this before?" (If yes) "When? What do you remember about it?" and so on. Interview 5 ends, like the others, with a retrospective discussion.

Selected interviews with the children form the basis of a number of studies. The thesis of Zang (1994) examines the development of strategic thinking in four of the children, comparing Interview 1 and Interview 4; the thesis of DeBellis (1996) studies affect in four of the children, using Interviews 1, 3, and 5; and the thesis of Passantino (1997) looks at the development of fraction representations for all of the children, comparing Interviews 2 and 5 (see also DeBellis & Goldin, 1997; Goldin & Passantino, 1996; Zang, 1995). With these scripts as examples, we now consider some general perspectives on structured, task-based interviews of this sort as a research technique in mathematics education.

ON THE SCIENTIFIC NATURE OF TASK-BASED INTERVIEWS

The longitudinal study, like many that use task-based interviews, is exploratory. Consisting as it does of a collection of individual case studies, its outcomes are not in a strict sense scientifically reproducible, and it might seem at first that this is all that can be said. Nevertheless, there are certain respects in which methods of scientific inquiry have been carefully regarded in the creation and administration of the interview scripts. I believe aspects such as these to be essential if we are to make real progress in understanding the nature of mathematical learning and problem solving through empirical observation. Thus, it is possible to envision the research being extended in a direction that permits replicability.

First, it is crucial to maintain carefully the scientific distinction between that which is observed and inferences that are drawn from observations. In this study we (at best) are able to *observe* children's verbal and nonverbal behavior, as captured on videotape during the sessions. From these observations, we (and others who use similar methods) seek to *infer* something about the children's internal representations, thought processes, problem-solving methods, or mathematical understandings. We cannot "observe" any of the latter constructs.

Second, our inferences are going to depend on (often tacit) models and preconceptions about the nature of what we are trying to infer and its relation to observable behavior. A scientific goal of the theory of mathematics education must be to make such models as explicit as possible. As we do this, we move away from depending on the ad hoc design of task-based interviews toward constructing them more consciously on the basis of explicit theoretical considerations. The task-based interview is like an instrument of scientific experimentation, and it is theory that describes how such an instrument is expected to interact with the system observed (in this case, the child as problem solver) so as to permit the drawing of valid inferences from the observations and measurements made. This point is discussed further in the next section.

Third, inferences from task-based interviews are likely to be unreliable, in that different observers may disagree about what inferences they would make after observing the same videotape—even when they agree on the theoretical constructs for which they are looking. The process of drawing inferences about children's thinking is fraught with uncertainty. At least at the outset, then, another scientific goal must be to describe the *criteria* that are to be used when inferences are drawn, so that the inferencing process itself becomes open to discussion.

For these issues to be addressed meaningfully, there must be a sense in which the task-based interviews are themselves explicitly characterizable as research instruments, subject to reuse, refinement, and improvement by different researchers. Thus in the study described here, we devoted great effort to structuring the interview scripts—ahead of their actual administration—to achieve two features: (a) flexibility and (b) reproducibility. Let us consider these twin goals.

Flexibility by the clinician in a task-based interview means being able to pursue a variety of avenues of inquiry with the learner or problem solver, depending on what takes place during the interview. Such flexibility is essential for our investigations to allow for the enormous differences that we know occur in individual problem-solving behaviors and that we infer exist in individual children's meaning-making activity. Because a major goal is to elicit and identify processes the children use spontaneously (i.e., without direct hints or coaching), flexibility is necessary to avoid "leading" the child in a predetermined direction in the problem solving.

Reproducibility in contrast means that the clinician is not merely inventing questions extemporaneously as the child responds. It permits, to a certain imperfect but improvable degree, "the same interview" to be administered by different clinicians to different children in different contexts. The degree to which this is possible increases as the research experience base with each particular interview accumulates. In asserting reproducibility as a fundamental goal, I am fully aware that I take a position at variance with the version of radical constructivism that asserts on a priori grounds its impossibility. The argument is sometimes made that since no two individuals ever solve "the same" problem, reproducibility is a fiction. The mistake of those asserting this position is confusing the problem instrument (the task, or task-based interview instrument, as structured by the researcher apart from the child) with the *interaction* being observed or measured (the problem solving that occurs when the child participates with the clinician in the actual interview). Of course, no two sequences of *interactions* are identical. From a scientific perspective, however, the wide differences that are observed to occur from interview to interview can be better understood and attributed when variables that are in principle subject to control (i.e., the task variables) are, in fact, controlled. Thus, the creation of reproducible task-based clinical interviews is an essential scientific step.

To accomplish this step, sufficiently many problem-solving contingencies must be anticipated. The criteria for the clinician's choices of questions or suggestions must be made as explicit as possible *in advance* for each contingency, with the balance covered by general instructions. This is what we have sought to do in the process of interview design.

For example, in Interview 1, three cards are presented. After a brief pause (to allow any spontaneous responses to the presented cards), the child is asked, "What do you think would be on the next card?" Contingencies then include "response" and "don't know." If the child responds, the next contingencies include "offers a complete, coherent reason" or "has not yet given a complete, coherent reason," with or without having constructed a "coherent external representation." The definitions (from the directions in the Interview 1 script) are as follows:

> A complete and coherent verbal reason means one based on a described pattern. A coherent external representation means a drawing, picture, or chip model. It is not required that the "canonical" fourth card (with 7 dots) be drawn, or the canonical pattern described, for a response to be considered a complete and coherent reason and a coherent external representation. An answer such as "7, because it's 2 more" is a coherent verbal reason, but it is not considered complete because it refers only to finding the next card and not to the basis for the pattern. An answer such as "7, because this card has 2 more than that one, so the next one has 2 more also" would be considered coherent and complete. If there is a discrepancy between the number of dots stated and the number in an external representation, the verbal reason is not considered "coherent." This [describes] the "boundary" between responses that are and are not accepted as complete and coherent....

The clinician's next question or suggestion (e.g., "Why do you think so?" or "Can you show me what you mean?" that leads, if necessary, to "Can you show me using some of these materials?") depends on the contingency that best describes the child's response. This is the level of detail at which many (although not all) contingencies are considered in the script design. We thus seek to make *explicit* the usually *tacit* conditions that ordinarily influence a skilled clinician. But this level of detail demands much preparation and rehearsal by the clinicians.

In principle, such detailed structured interview descriptions lead to several desirable features: (a) increased *replicability* of the interview itself, although contextual and other factors will still vary widely from occasion to occasion, and, of course, the knowledge structures of individual children are highly variable; (b) a degree of *comparability* of interview outcomes between different children, across different populations of children, across different conditions of school learning, and so forth; (c) subsequent experiments to investigate the *generalizability* of observations made in individual case studies; (d) explicit *discussion and critique* of the contingencies built into the interview, which permits the criteria for the clinician's responses to be analyzed and improved; and (e) an explicit basis for discussing the analysis of outcomes, that is, the process of drawing inferences from observations. For earlier perspectives on these ideas, see Cobb (1986), Goldin and McClintock (1980), Hart (1986), and Steffe (1991a).

THE ROLE OF THEORY

One purpose of clinical task-based interviews in mathematics education is to permit us to characterize children's strategies, knowledge structures, or competencies—perhaps to be able to look at the effectiveness of instruction, to understand developmental processes better, or to explore problem-solving behavior. However we choose to define our inferential goals, a theoretical framework for describing or characterizing what we seek to infer is necessary. But the role of theory is not limited to this. Theory must also tell us something about how the characteristics of the task in the task-based interview (e.g., its language, its mathematical content and structure, its appropriateness for particular cognitive processes, the interview context) are expected to interact with the cognitions we are trying to infer, so that the interview can be designed to elicit processes of the desired nature. To say that the problems in the task-based interviews described here are of a level of complexity thought to permit a variety of strategies to be employed, or internal representations to be constructed, already presupposes major theoretical assumptions.

The questions asked and the observations made during any scientific investigation, including investigations using task-based, clinical interviews, depend heavily on the theory we bring to it. Thus, in my view, the main question is not whether theory *should* influence us in this enterprise. I maintain, in agreement with R. B. Davis (1984), that it always, inevitably does:

> Perhaps the attempts to use the methods of science [in education] have failed because science has been misunderstood.
>
> In these attempts it had been assumed that science was primarily factual, that indeed it dealt almost solely in facts, that theory had no role in science. Careful observation of science reveals this to be false. It might be closer to the truth to say that "facts"—at least interesting facts—are almost unable to *exist except in the presence of an appropriate theory* [emphasis in original]. Without an appropriate theory, one cannot even state what the "facts" are. (p. 22)

The question pertaining to clinical interviews is the extent to which the influence of theory remains tacit, taking place through *unconscious* assumptions of clinicians, researchers, and/or teachers, or becomes explicit and thus open to discussion and challenge. Our goal in the present study is to be as explicit as possible.

The theoretical underpinning of this series of interviews includes the concept of (internal) *competencies* and structures of such competencies. These are envisioned as developing over time in the child and as being capable of being *inferred* from observable behavior—when the appropriate conditions exist for the individual to take certain cognitive steps and some corresponding behaviors are seen. Another fundamental theoretical assumption is the idea that competencies are encoded in several different kinds of internal representations and that these *interact* with one another and with observable, external representations during problem solving. A third assumption is that *representational acts* occur in which representational configurations (internal or external) are taken to symbolize or stand for other representational configurations.

The model that most strongly influenced the development of the scripts is one that I have been developing for some time as a way of characterizing mathematical problem-solving competency. It includes five kinds of mutually interacting systems of internal, cognitive representation (Goldin, 1987, 1992b): (a) a verbal/syntactic system (use of language); (b) imagistic systems (visual/spatial, auditory, kinesthetic encoding); (c) formal notational systems (use of mathematical notation); (d) planning, monitoring, and executive control (use of heuristic strategies); and (e) affective representation (changing moods and emotions during problem solving). The interplay between the children's internal representations and external representations that they use or construct during the interviews provides one of the most important means of drawing inferences.

For example, from children's descriptive statements about what a birthday cake looks like (Interview 5, Problem 3) we infer internal visual/spatial representations. From their gestures as they describe how they would cut a birthday cake into 2 or 3 pieces (Interview 3) or 12 pieces (Interview 5), with accompanying drawings, we infer simultaneous internal, kinesthetic representations. Children's explanations of the fractions written symbolically in Interview 5 permit inferences concerning their internal representations of this formal mathematical notation. Steps they take relating one sequence of cards to another in Interview 4 permit inferences concerning internal executive control (heuristic or strategic representations). Affective representation is inferred not only from the child's statements in response to questions, but also from facial expressions and spontaneous comments and gestures. I would stress again that the whole process of inferencing is, at this stage in the research, of limited reliability, but that strengthening the degree of reliability is an important goal.

Since the study is longitudinal, a major focus is how systems of representation *develop* in the child over a period of time. In this respect, the theoretical model incorporates three main stages: (a) an inventive/semiotic stage, in which internal configurations are first assigned "meaning," (b) a period of structural development, driven by the meanings first assigned, and (c) an autonomous stage, in which the representational system functions flexibly and in new contexts. We hope to be able to infer representational acts associated with each of these stages.

The distinction between external and internal representation means that we must attend carefully to both. We regard the tasks posed as *external* to individual children, embodying syntax, content, context, and structure variables that we select when we design the interviews. In particular, the mathematical structures of the tasks (semantic structures and formal structures—additive, multiplicative, and so forth) are consciously chosen. The behaviors observed result from *interactions* between the task environment and the child's internal representations.

To posit interactions between internal and external representational systems thus requires a great deal of analysis of mathematical structures associated with the tasks. Parallel but not identical structures—in some instances, homomorphic structures, in other instances, structures less directly related—were intentionally included in the different interviews. For example, a certain additive structure is embodied in the (canonical) sequence in Interview 1. Other additive and multiplicative structures relate to the sequences in Interview 4, which are also structurally related to each other. The card sequences are all presented in parallel ways to the children. A certain multiplicative structure underlies the cube-cutting task in Interview 2. Reflection symmetries are embodied in the cards in Interviews 1 and 4, in the cutout and cube-cutting tasks in Interview 2, and in the birthday cake task in Interview 3. More subtle, hidden symmetry is present in the jellybean problem in Interview 3. Rational number structures occur in Interviews 2 and 5. The analysis of all these relationships is theory based, and many assumptions are being made just in asserting that a structural relationship between tasks exists.

Another key theoretical distinction is between the child's *spontaneous* bringing to bear of any particular competency, or the child's doing so only when *prompted.* This is a subtle but crucial distinction, which involves the child's exercise of planning competencies to call on other competencies (verbal, imagistic, formal notational). For example, from a child's spontaneous response to the task in Interview 1 that each card is two more than the previous card can be inferred the implementation of at least part of a problem-solving plan. Should the child make the same observation only after being asked by the clinician, inference of such a planning representation would be unwarranted. These ideas have influenced the task-based interview development as follows: We pose tasks that permit the children to perform at each step *spontaneously.* We explore not only the child's overt behavior, but the *reason* the child states for taking each step. Recognizing that competency structures may be partially developed, we do provide hints or heuristic suggestions when blockage occurs. This often permits the child to demonstrate competencies that otherwise he or she would never "get to" during the problem solving, which adds to the information gained. There is always a trade-off entailed here, in that the more specific the hint or suggestion provided by the clinician, the less extensive the information gained about the child's representation of planning and executive control in problem solving.

We seek information about each kind of internal representational system; thus, not satisfied with a coherent verbal explanation only, we nearly always encourage the child to construct a concrete, external representation. We include a cross section of questions exploring visualization, affect, and strategic thinking. In particular, Interview 2 is designed especially to detect and explore in greater depth imagistic systems (visual/spatial and tactile/kinesthetic) in problem solving while attending to affect and to other kinds of internal representation; Interview 3 focuses on affect in greater depth (see also McLeod and Adams, 1989), whereas Interview 4 returns to tasks selected for the possibility of eliciting certain planning or strategic representational capabilities.

It is my view that the characteristics of the task-based interviews are variables that are *inevitably* built into clinical interview designs. Considerations of task structure in these interviews are sufficiently complex to form the basis themselves of several articles, yet task structure is an essential component to understanding and making inferences from observed problem-solving behavior. It needs to be examined independently of the individual children as a part of the process of drawing inferences from children's interactions with the tasks. My main point is that there is no way to avoid this interplay between theory and observation. It is not a sufficient answer to respond, as some do, that task structures do not "exist" apart from individual problem solvers. *We simply have the choice of proceeding unscientifically, choosing tasks that seem interesting and just "seeing what happens," or trying to proceed systematically with tasks* *explicitly described and designed to elicit behaviors that are to some degree anticipated.*

Although the analysis of outcomes in these interviews is theoretically based, we seek not only to observe and draw inferences from *expected* processes but also to search for unanticipated occurrences. The hoped-for results include the further refinement and development of the theoretical model for problem solving, including the identification of inadequacies and progress toward an assessment framework, as well as conjectures for further investigation through future experimental studies.

THE ROLE OF CONTEXT

Task-based interviews do not take place outside of a social and psychological *context.* That context influences and places constraints on the interactions that occur during an interview and puts limitations on the inferences that can be drawn. It is one of the components that theory must address, if we are to validly interpret interview outcomes.

The view taken here is that "social and psychological context" affects the interview interactions through internal representations that the child has constructed, which are in principle subject to description. These are considered "contextual" because the semantic content of the representational systems involved is not, at least initially, mainly derived from, or related to, intended mathematical representations associated with the tasks posed in the interviews.

We observe, for example, that the child's expectations in an interview may be influenced by the fact that it is conducted by a relative stranger, the clinician. The interview takes place in school and thus might be assumed by the child to involve some kind of test that "counts" toward an evaluation. Children often seem to think, especially at the outset, that the tasks are likely to have "right" and "wrong" answers and that certain methods will meet with the clinician's approval, whereas others will not. The interview itself may be taking place at a moment when the child is alert, tired, hungry, distracted, or excited. On the one hand, the child might prefer to be back in his or her regular class with friends or might, on the other hand, be looking forward to an interesting break from the classroom routine. The fact of being videotaped was for the children in our study a familiar experience (owing to the project in which their teachers were participating); the context of their experience would be different were the video cameras a complete novelty.

It seems to be an almost inevitable feature of task-based, clinical interview methodology that the tasks are unrelated, at least initially, to a goal or purpose generated by the child. For example, the butter problem and the toymaker problem (Problems 2 and 4 in Interview 5) are both posed in a *stated* context. The butter problem (or one like it) is a problem that could conceivably arise as a practical need in a variety of real-life situations not too different from the stated context. It would very likely be experienced differently if the child were actually *in* one of those situations and had generated the problem goal (as opposed to solving the problem as part of a clinical interview). The toymaker problem, in contrast, is a rewrite of a rather well known mathematical problem involving married couples in a village. We rewrote the problem to present a concrete, external representation with which the child could experiment if desired. Although the context of making toys is one the child can easily imagine, the problem goal is not one that occurs "authentically" in that context. It is posed as an almost whimsical question, arising perhaps as a curiosity (curiosity-based problem solving is, of course, an essential aspect of mathematical inquiry) but not as a practical question that needs to be answered for the toymaking to proceed. Thus, the contexts of these two problems are different in an important respect. Such contextual factors could influence, for example, the importance that the child ascribes to the problem goal and, in turn, the child's persistence, enthusiasm, choice of strategy, and so forth.

Another meaning of context, one that might be called "mathematical context," refers to unstated aspects of the tasks themselves as they are presented during the interview—aspects that although seemingly small may have important effects. For example, in presenting the three cards in Interview 1 and again (several times) in Interview 4, we permit the child to see the cards being drawn from a stack of cards in a manila envelope. From this minor contextual feature (which was intentionally included), the child may infer that there is a deck of cards larger than the three that are shown and, possibly, that there is a pattern in the cards. Three cards presented wholly out of context might not so readily elicit this expectation. Evidently, certain contextual influences are undesirable (e.g., those that might mask our ability to observe competencies that are present in the child), whereas others are helpful (e.g., those that would facilitate the child's "thinking mathematically").

Since so much that may occur during a task-based interview is context dependent, how can we consider what we observe to be more than accidental, one-time events? One important condition is to require that *the constructs we infer from our observations be reasonably stable against contextual variations*. For example, suppose we infer, in Interview 2, a child's ability to represent imagistically (visually, kinesthetically, or both) the cutting of a cube across two perpendicular directions. The inference may be drawn from the child's coherent description of the component pieces of the cube, with appropriate gestures indicating how the cube was imagined to be cut. Although it is indeed the case that this child's behavior may vary considerably from one context to another, when we infer such particular competencies or structures of competencies from that behavior, we are inferring aspects of the child's cognition that we expect to be fairly stable. If the inferred competency were to disappear in short order, it would not be useful in a theory of mathematical learning.

Understanding the contextual dependence of the interviews also means recognizing how very difficult it is to establish advance criteria for all the inferences about each child's cognition and affect that we want to draw from our observations. When observations are interpreted in context, new likelihoods occur. The plan we have been following is to make the best conjectures possible and to try to be explicit about the reasons for conjectures, including relevant contextual factors, as these occur (Zang, 1994).

Such discussions of contextual issues barely scratch the surface. For task-based interview methodology to be pursued seriously, a deeper understanding—indeed, a theory of how social, psychological, and mathematical contextual factors may influence mathematical problem solving during a task-based interview—is essential to the interview design process.

PRINCIPLES OF INTERVIEW DESIGN

I conclude this chapter by summarizing what, in my opinion, are some of the most important underlying characteristics of the five interviews described here and try to abstract from these the most salient general principles behind their design. Although each interview has its own particular focus, certain broad characteristics are maintained in all of them:

1. Each interview is based on particular mathematical ideas appropriate for the age group of the children (grades 3–6) and on mathematical topics with associated meaningful, semantic structures, as well as formal, symbolic structures, for example, additive or multiplicative structures, sequences, schemata underlying rational number concepts, and so forth. We want the mathematical content to be based on topics that can be studied in depth and are flexible enough to allow evidence of widely differing capabilities on the part of the students.

2. Each interview consists of a series of questions posed in one or more task contexts. These begin at a level that all the children are expected to understand (of course, in differing ways). They become increasingly difficult, culminating in questions that can still be attempted by all the children but that will pose major challenges even to the most mathematically astute students.

3. The children engage in *free problem solving* to the maximum extent possible. This prioritizes exploring the strategies that the children use spontaneously—whatever method or methods seem most appropriate to them as they work on the task. They are reminded occasionally to talk aloud about what they are doing and to describe what they are thinking. Hints and prompts, or new questions, are offered *only after* the opportunity for free problem solving and are then followed by a further period of observing how the child responds without directive intervention. This rule is (in view of time constraints) occasionally broken because of our desire to ensure reaching a subsequent section of the interview in the allotted time, but it is broken with the recognition that possibly important information is necessarily being lost.

4. All student productions are "accepted" during the interview; the clinician does not impose preconceived notions about appropriate ways to solve the problem but does treat "wrong" answers similarly to "correct" answers (with occasional,

specified exceptions). Responses elicit follow-up questions without an indication of correctness. The rare exceptions, involving guiding the students toward particular understandings, are decided in advance and occur only where the understandings are essential for subsequent interview questions to be meaningful.

5. Materials for constructing a variety of external representations are available for student use and vary from task to task: paper and pencil, markers, cards, chips and other manipulatives, paper cutouts, hand calculators. A major task goal is always the construction of representations by the children—ideally, a multiplicity of them.

6. Each interview includes reflective questions, typically posed retrospectively, that address the child's problem-solving processes and the child's affect.

7. Because the interviews are designed for use in a longitudinal study, there is a conscious effort to incorporate into later interviews some tasks that are similar in context, mathematical content, structure, or all three, to those posed earlier.

Building on these specific characteristics and the issues discussed in this chapter, I propose to formulate the following tentative and partial principles of interview design and construction with the goal of trying to establish the strongest possible scientific foundation and maximizing the information gathered through a task-based interview.

1. *Accessibility.* Interview tasks should embody mathematical ideas and structures appropriate for the subjects being interviewed. Subjects must be are able to represent task configurations, conditions, and goals internally and, where appropriate, externally.

2. *Rich representational structure.* Mathematical tasks should embody meaningful semantic structures capable of being represented imagistically, formal symbolic structures capable of notational representation, and opportunities to connect these. Tasks should also suggest or entail strategies of some complexity and involve planning and executive-control–level representation. Opportunities should be included for self-reflection and retrospection.

3. *Free problem solving.* Subjects should engage in free problem solving wherever possible to allow an observation of spontaneous behaviors and reasons for spontaneous choices. Providing premature guidance results in a loss of information. This principle may mean some sacrifice of the speed with which the subject understands the problem or progresses through it.

4. *Explicit criteria.* Major contingencies should be addressed in the interview design as explicitly and clearly as possible. These contingencies should distinguish "correct" and "incorrect" responses (but rarely) with structured questions designed to give subjects opportunities to self-correct in any contingency. This is an important key to the replicability and generalizability of task-based interview methodology.

5. *Interaction with the learning environment.* Various external representational capabilities should be provided, which permits interaction with a rich, observable learning or problem-solving environment and allows inferences about problem solvers' internal representations.

It is hoped that the discussion in this chapter furthers the goal of understanding mathematical learning and problem solving scientifically through the use of task-based interviews as research or assessment instruments.

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