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ARTICLES

Advanced Mathematical Knowledge in Teaching Practice: Perceptions of Secondary Mathematics Teachers

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For the purpose of our research we define Advanced Mathematical Knowledge (AMK) as knowledge of the subject matter acquired during undergraduate studies at colleges or universities. We examine the responses of secondary school teachers about their usage of AMK in teaching. We find that the majority of teachers focus on the purposes and advantages of their AMK for student learning, such as personal confidence, the ability to make connections, and to respond to students' questions; only a few provide content-specific examples. We conclude with a call for a more articulated relationship between AMK and mathematical knowledge for teaching.

THE GAP BETWEEN UNDERGRADUATE AND SECONDARY MATHEMATICS

This study is motivated by the observation of significant gaps between secondary school mathematics and tertiary mathematics. Students, even those identified in school as high-achieving students, experience unexpected difficulties when beginning undergraduate mathematics courses, and many teachers perceive their undergraduate studies of mathematics as having little relevance to their teaching practice. The latter issue is our interest in the current research.

To close or, at least, narrow these gaps, the Conference Board of the Mathematical Sciences (CBMS) report on the mathematical education of teachers (2001) offered two suggestions about what mathematical knowledge teachers should entail and how it can be acquired. First, they recommend that core mathematics courses should be redesigned to help future teachers make

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insightful connections between the advanced mathematics they study in university and the secondary school mathematics they will be teaching. Second, they recommend that a "capstone" course that examines fundamental issues in the secondary mathematics curriculum from the standpoint of advanced mathematics be developed. While a conversation of what background is required for teaching secondary school mathematics continues, a recent report from the U.S. National Mathematics Advisory Panel concluded that there is a dearth of solid research on how teachers use their mathematical knowledge in helping students learn (Mervis, 2008; National Mathematics Advisory Panel, 2008). Teachers' usage of mathematical knowledge in their teaching of secondary school mathematics is the focus of our investigation.

In this article we first introduce the notion of Advanced Mathematical Knowledge (AMK) and situate it within the ongoing discussion on Advanced Mathematical Thinking. Next, we consider research on teachers' knowledge or knowledge for teaching, and discuss the relationship (or lack thereof) between teachers' knowledge of mathematics and the achievements of their students. We then turn to the results of our study and analyze the views of secondary school mathematics teachers related to their usage of AMK in their teaching practice.

THEORETICAL CONSIDERATIONS

Two strands of research in mathematics education influence our theoretical position: research on advanced mathematical thinking and research on teachers' knowledge. We provide a brief review of each strand in order to articulate our perspective.

AMK and Advanced Mathematical Thinking

For the purpose of the study we define Advanced Mathematical Knowledge (AMK) as knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college. (We acknowledge that existence of different curricula makes our definition of AMK time and place dependent; however, sufficient similarities among the curricula make it useful for our pursuits.) We define teaching practice broadly to include lesson planning, student assessment, instructional materials preparation, in-class presentations, interactions with students, and reflection.

AMK is associated with the notion of Advanced Mathematical Thinking (AMT), which is receiving continuous attention in mathematics education. The seminal volume *Advanced Mathematical Thinking* edited by David Tall (1991) was a landmark that positioned AMT as an area of research in mathematics education. It also intensified conversations on what constitutes AMT, and how it can be identified and supported.

In attempts to define AMT there was no agreement among researchers on whether the adjective "advanced" described advanced mathematics or advanced thinking in mathematics. Harel and Sowder (2005) highlighted this tension by relocating the hyphen, posing "advanced-mathematical thinking" (i.e., thinking in advanced mathematics) versus "advanced mathematical-thinking" (i.e., mathematical thinking of an advanced nature). The difference in perspectives on what constitutes AMT shifted the focus, or at least the description of the research area, from AMT to tertiary mathematics (Selden & Selden, 2005). As such, our definition of advanced mathematical knowledge (AMK) accords with this shift: AMK is knowledge related to topics in tertiary mathematics. The purpose of this study is to explore teachers' conceptions of the role and usage of AMK in their teaching practice.

Teachers' Knowledge and Teaching Practice

While teaching is unimaginable without teachers knowing the subject matter, it is unclear how "knowledge *for* teaching" can be measured. The most used measure, the number of mathematics courses taken by a teacher, did not lead to conclusive results. Begle (1979) found that students' mathematical performance was related neither to the number of university courses their teachers had taken nor to teachers' achievement in these courses. However, Monk (1994) found a minor increase in secondary students' achievement associated with the number of college courses in mathematics taken by mathematics teachers. Further, "researchers at the National Centre for research on teacher education found that teachers who majored in the subject they were teaching often were not more able than other teachers to explain fundamental concepts in their discipline" (NCRTE, 1991, quoted in CBMS, 2001, p. 121).

More recent studies recognized the inherent complexities with these kinds of results; mainly that the degree held and number of courses taken by a teacher are not appropriate measures of mathematical knowledge. Following a comprehensive literature review, Hill, Rowan, and Ball (2005) concluded that measuring teachers' mathematical knowledge more directly by looking at scores on certification exams or exam items related to a specific topic generally revealed a positive effect of teachers' knowledge on their students' achievement.

Struggling with the question of what kind(s) of teachers' knowledge benefit teaching and learning, researchers realized that mathematics knowledge *for* teaching (Ball, Hill, & Bass, 2005), or simply "mathematics-for-teaching" (Davis & Simmt, 2006), may be a special "register" of knowledge, a special combination of content and pedagogy, that relies on deep understanding of the subject and awareness of obstacles to learning. This specialized knowledge has received some attention at the elementary level (e.g., Ma, 1999), and it has been shown that such specialized knowledge for teaching was significantly related to students' achievement at elementary grades (Hill, Rowan, & Ball, 2005). At the secondary level the issue of teacher's knowledge has yet to be explored in detail. Our study stems from a position that engaging with advanced mathematical content (that is, AMK) is a necessary (although not a sufficient) condition for achieving this specialized knowledge for teaching at the secondary level.

THE STUDY

As mentioned previously, we were interested in how secondary school teachers of mathematics articulate their usage of AMK by describing specific instances of such usage. Our general research question was:

- What are teachers' perceptions of their usage of AMK in their teaching?

More specifically we investigated the following questions:

- How do teachers perceive and describe the extent of their usage of AMK in their teaching?

- How do teachers perceive and describe the ways in which their AMK is implemented in their teaching?
- What are specific examples that teachers provide to illustrate their usage of AMK in teaching mathematics?

Participants and Data Collection

The 52 participants in this study were practicing secondary school teachers, teaching Mathematics in grades 8–12, including Algebra, Geometry, and Calculus. They were either enrolled in various professional development courses at the time of the study or recently completed such a course. The background of participants varied significantly, from individuals holding Bachelor's degrees in Mathematics or Master's Degrees in Mathematics Education to teachers whose formal education was in Sciences or Engineering.

In our previous interactions with teachers that included informal conversations or interviews we asked teachers to reflect on their teaching and to share experiences in which they used their AMK. These usually resulted in general claims rather than particular examples. The teachers claimed that it was hard to come up with examples "on the spot." Acknowledging this difficulty, and because of the vagueness of some responses, we designed and implemented a written questionnaire that attempted to elicit specific and detailed responses. It consisted of the following questions:

1. To what extent are you using AMK in your school teaching?
2. Provide an example (and if possible several examples) of mathematical topics from the curriculum in which, in your opinion, AMK is essential for teachers. In each topic specify the usage of AMK.
3. Provide an example (and if possible several examples) from your personal experience of a teaching situation (such as classroom interaction, preparing a lesson, checking students' work, etc.) in which you used AMK. Provide detailed description of each case.
4. Provide an example (and if possible several examples) of mathematical problems or tasks from the school curriculum in which AMK is necessary or useful for a teacher. In each case describe the usage of AMK.

We note that questions 2, 3, and 4 are related; for example, a teaching situation sought in question 3 can be connected to a topic in question 2 or a problem in question 4. However, a potential redundancy was intentional in order to solicit further responses. The time for completing the questionnaire was not limited and the teachers could consult any resources they found appropriate. The questions were preceded with a definition of AMK consistent with our working definition mentioned above:

> In this questionnaire we refer to knowledge of the subject matter acquired in Mathematics courses taken as part of a degree from a university or college as "Advanced Mathematical Knowledge."

We provided teachers in advance with questions they were asked to address. Considering the effort involved in completing this questionnaire, we presented participants with the opportunity to respond orally in a clinical interview with a researcher or research assistant. Ten teachers preferred this approach. Although the questions were predetermined, the interviews were carried out in a semi-structured way, allowing for further prompting and clarification of responses. Each interview lasted about 30 minutes, and teachers' responses were audio recorded and transcribed. As such, our data is based on 42 written responses and 10 individual interviews.

Data Analysis

The data analysis was ongoing over the period of data collection and recursive; that is, the ideas that were generated through the analysis of initial data were further examined by the analysis of additional data as well as repeating the analysis of the same data by both researchers. This process is consistent with the qualitative approach based on grounded theory procedures and techniques (Strauss & Corbin, 1990).

Despite a significant variety in participants' background, there appeared to be no apparent relationship between their academic education, teaching experience, or grade level teaching assignment and teachers' claims about their usage of AMK. As such, investigating these correlations was not a focus of our current study.

In our initial consideration of the data we attended to three themes that emerged in a previous study that explored the usage of AMK with six teachers (Zazkis & Leikin, 2009): connection to the history of mathematics, meta-mathematical issues (by "meta-mathematical" we mean cross-subject themes, such as definition, proof, example, counterexample, language, elegance of a solution, etc.), and mathematical content. However, a much larger number of participants in the current study presented a greater variety of responses, some of which did not fit within the previously identified themes. As such, these themes have been refined, and additional persistent themes have been identified as related to pedagogy, teachers' confidence, and connections within and beyond curriculum. Although the questionnaire had four questions, most participants did not differentiate among their answers to Questions 2, 3, and 4 in the written responses as well as in the interviews. The emergent themes arose through content analysis of teachers' replies across the four questions in the questionnaire.

RESULTS

In presenting the data our goal is to demonstrate the emerging themes. However, we note occasionally the number of responses in order to exemplify persistence of a certain theme.

We present the emerging themes in three categories: (1) Frequency/extent of usage of AMK, (2) Ways of usage, and (3) Purposes/advantages of usage. Frequency or extent of usage relates specifically to Question 1 in the written questionnaire. When responding to Questions 2, 3, and 4 the participants had a tendency to describe how and when their AMK is used and the purposes or benefits of this usage. These descriptions appeared either in addition to or instead of the specific detail in situations or problems that was sought. As such, the categories (2) and (3) are interrelated as the purpose of application often shapes the ways in which knowledge is applied.

Frequency/Extent of Usage of AMK

The participants' claims varied significantly in responding to Question 1: to what extent they used AMK in their school teaching. Their responses indicated their perceived frequency of usage and also the relevance to their practice. The 42 written responses can be clustered in four groups: “not essential” (6 responses), “rarely” or “very little” (16 responses), “it depends” (13 responses), and “all the time” (7 responses). The responses from the interviews are not included in this count, as following interviewer’s prompts the participants may have changed their minds several times. In what follows we illustrate and discuss teachers’ responses in each cluster.

“Not essential/not relevant”

The written responses of 6 (out of 42) teachers explicitly stated that AMK was not essential and was not used in their teaching. The following claims exemplify this perception:

> I don’t think I use any AMK in my teaching. I would say that AMK is non-essential. (Nick-1)
>
> I am able to teach secondary school mathematics without AMK. (Annie-1)

Teachers who made these claims felt the need to elaborate on their background:

> I could do/teach everything in the Math 8–12 curriculum before going to university as I excelled in High School Math and have tutored over the years. (Nick-2)
>
> I do not think my students are at a disadvantage because I do not have AMK. In fact, I met a former student of mine at the graduation ceremony. He introduced me to his family and friends as the best math teacher he ever had. AMK is probably not necessary for teaching secondary school math since I managed so many years without it. (Annie-2)

However, when a similar claim appeared in a clinical interview, the interviewer “pushed” and attempted to clarify:

Interviewer: So, would you say that your university training was completely irrelevant?
Lisa: No, it wasn’t irrelevant, but it wasn’t relevant either.
Interviewer: So what parts, if any, were relevant?
Lisa: If we are talking about content only, I would say that is all irrelevant. It is more just a way of thinking, analytical thinking and the concept of proof. Working towards a goal and not giving up. [pause] I would say that what I bring to high school teaching from university is a good work ethic and discipline.

In this excerpt Lisa diverts our attention from the mathematical content and mentions a way of thinking and the concept proof. Although it would be impossible to develop these without exposure to any content, Lisa’s view suggests that content is considered as means to another end rather than as a goal. This idea is revisited in the subsequent sections.

“Rarely” or “very little”

The responses in this category—provided in 16 out of 42 written questionnaires—echo the sentiments of those in the previous section, with a slightly different wording. The following claims illustrate this kind of response:

> The bottom line is that very little[1] of anything you learn in university is directly useful for teaching high-school math. However, some of what you learn will help give you deeper understandings and find connections in math, thus improving your ability to enrich the students experience while teaching math. I very easily could have taught high-school the year after I graduated from high-school. (David-1)
>
> I don't believe I use AMK often (or at least not consciously) in my school teaching, as I have forgotten much of what I learnt during my undergrad. However, while I don't remember much of the content that I should have acquired during my undergrad, I feel that the courses I took nonetheless developed my ability to problem solve and think about high school mathematics to a greater depth. (Hanna-1)
>
> Not much. I am, however, aware of what the university math courses demand and make sure my students are prepared for their subsequent math courses. (Terry-1)

The repeating pattern in these responses is that the claims of "very little" or "not much" are followed by the conjunction "however" and an example of where AMK is found applicable. The themes identified here are deeper understanding and improved teaching (David), problem solving ability (Hanna), and preparation for the subsequent courses (Terry). As such, while these (and other) teachers do not find a direct and explicit applicability of AMK, they acknowledge indirect benefits. In fact, as is shown below, a majority of responses fall into this latter category.

"It depends"

About a third of written responses—13 out of 42—describe the extent of usage of AMK as dependent on a specific content. The following responses illustrate the theme and identify the places in which AMK is considered helpful.

> Although AMK, at times, proves to be very useful in mathematics classroom discourse, I have not been able to use it directly on a regular basis. It all depends on the topics and the nature and direction of the classroom discourse. (Kris-1)
>
> I certainly use the knowledge I acquired in Calculus courses to teach Calculus. While most of the other courses I teach don't seem to draw directly from knowledge I acquired at the post-secondary level, they do draw indirectly from it. To a certain extent, the AMK which is essential for teachers depends on which courses the teacher is teaching. (Molly-1)
>
> The actual theory acquired at university has only been used in the more senior courses. Again this is seldom applied in the regular classroom setting, but the AMK would be useful for teachers sponsoring Math clubs or teaching enriched or honours courses. (Sandy-1)

As exemplified in these responses, teachers consider AMK applicable for teaching senior level courses, especially Calculus, as well as enrichment lessons or courses. The apparent assumption in these responses is that of direct applicability. That is, the applicability of AMK to teaching practice is acknowledged if the teacher is teaching in school the same topic that s/he learned at a University.

[1] Some expressions in the provided excerpts are underlined by the authors for emphasis.

"I use it all the time"

Seven teachers made this or a similar claim.

> I use it all the time. (Daniel-1)
>
> I have found that I use AMK a lot in my teaching, from interjecting advanced or interesting "pieces" to full blown lessons based on what I have learned. I would say that rarely a day has gone when I have not used AMK. (Ken-1)

However, claiming that AMK is used or not used did not correspond to exemplifying such usage with detailed descriptions. Out of the seven teachers who claimed using AMK "a lot" or "all the time," only three provided detailed and specific mathematical examples of such usage.

Ways of Usage

In this section we address teachers' descriptions as to when, where, and in what ways their AMK is applied. We first address issues related to mathematical content: we mention mathematical topics identified by teachers and then illustrate general or specific situations or tasks by which teachers exemplified their AMK usage. We then turn to issues that we identified as meta-mathematical; that is, issues that can be integrated with a variety of topics.

Mathematical Topics and "General" Examples

In responding to Question 2—asking to provide examples of mathematical topics from the curriculum in which AMK is essential for teachers—most topics that participants mentioned related to Calculus. Teachers mentioned definition and usage of derivative, limits, and asymptotes. These topics further featured in teachers' examples provided in response to Questions 3 and 4. This is hardly surprising, as the topics of Calculus are the last ones that are usually learned in high school and the first ones encountered in undergraduate studies of mathematics. In fact, in 28 written questionnaires and 6 interviews Calculus was mentioned explicitly.

The second popular area in which AMK was perceived useful was Probability and Statistics. This is likely due to the fact that these topics were introduced to many secondary school curricula only recently, and the teachers had only very limited exposure to these topics in their schooling. (Some of the Probability and Statistics related topics would not be considered "advanced" according to our definition, to a more recent high school graduate.) To a lesser degree there was a mention of topics related to Linear Algebra (matrices, systems of equations), Number Theory, and Combinatorics. Again, these choices are related to direct applicability of what was learned to what is taught.

Other topics mentioned related to extra curricular enrichment activities and preparation for the contests, as illustrated below:

> I have sometimes used modular arithmetic when helping students prepare for the Math contests. I used optimisation and scheduling activities with an enriched class. We looked at the 4 colour problem in the same enriched class. (Selina-1)

> Though it is not part of the curriculum sometimes I entertain my students (at any level) with ideas from number theory (mostly code theory and RSA encryption) in telling them how math is used in the "real world." (Darlene-1)
>
> I mostly use AMK to engage the student in mathematics (although what I may talk about may not be explicitly a part of the curriculum). For example, in my Grade 12 class, when discussing sequences and series, I find that many of my students enjoy "off-topic" discussions about Fibonacci. (Maya-1)

Although Questions 3 and 4 on the questionnaire asked explicitly for detail, this request was ignored by about one half of the respondents. Only 12 teachers provided specific examples of mathematical problems or situations. Eleven teachers provided examples that we consider "general." The most popular example related to the use of Calculus in analyzing and sketching the graphs of rational functions, determining the points of minimum and maximum and the points of inflection, and in considering asymptotes in connection to limits and point of discontinuity. Additional advantage of Calculus knowledge was mentioned by Molly:

> The study of calculus gives the teacher a deeper understanding of the relationship between surface area and volume. For example, the derivative of the volume of a sphere formula gives the surface area of a sphere formula. In fact this knowledge helps me "remember" the formulas. (Molly-2)

Another repeating example (mentioned by nine teachers) was that of complex numbers. Teachers mentioned that their familiarity with complex numbers from their university courses helps in explaining why some quadratic equations "have no solutions," and emphasized that a more appropriate description is to say that there are "no real number solutions" rather than "no solutions." Three teachers mentioned that AMK was useful in teaching logarithms and introducing the transcendental number e; however, specific detail of this usefulness was omitted.

Within issues related to mathematical content we further differentiated between (a) responses that identified mathematical tasks or situations that clearly related AMK to secondary school curriculum, (b) responses that described tasks requiring AMK but not related to curriculum, and (c) descriptions of complicated tasks or problems with solutions based on the mathematical content from the secondary school curriculum rather than AMK. In what follows we exemplify each category with illustrative examples.

Examples Related to School Curriculum and AMK

In planning our study we were most interested initially in examples in this category. The specific questions invited responses related to the topics taught in secondary school. As mentioned previously, teachers' difficulty in generating examples "on the spot" led us to design questions for a written response and present these questions in advance for those choosing the interview. However, despite unlimited time, only a limited number of examples were explicitly related to the secondary school curriculum. Several examples presented below introduce different ways in which AMK is used.

In her interview Rachel described that when working with low achieving students on solving a system of two linear equations, she wanted the results to be integers. To achieve this, without building the equations by substituting the solutions, she relied on her knowledge of determinant and inverse matrix algebra, acquired in a Linear Algebra course. She showed that when the determinant is 1 or (–1) the values of unknowns are integers. She exemplified this using the parametric form of equations:

> If ax+ by = c and dx + ey = f, then x = (ec – fb)/(ae – bd) and y = (fa – cd)/(ae – bd)
> As such, in building equations she chose det [ ] = ae – bd = ± 1. (Rachel-Interview)

Pat recalled that when the task was to find the coordinates for the vertex of a parabola, her students, not exposed to Calculus, had to find the roots of the related polynomial, where the midpoint between the roots was the x-coordinate, and then use the equation for a parabola to find the y-coordinate. She could quickly check their solution using Calculus, finding the derivative and, with the help of derivative, finding the extremum point. (Pat-1)

Kris mentioned that he was often asked by his students whether there was any general formula to generate Pythagorean triples. In his response he provided the algorithm and proved it. He further noted that this knowledge assisted him in generating tasks for students:

> Knowledge about how to generate the triplets, at times, proves to be very useful in solving problems related to right triangles/Pythagoras theorem. However, the high school curriculum does not either provide the general formula to generate the triplets or if it does, it lacks to explain how and why the formula works. So, I found my AMK very helpful in teaching this topic. Further, I used to generate Pythagorean triples in making up questions for students where they have to apply Pythagorean theorem. Even today, when everyone has a calculator, it is nice to get a whole number in the answer. (Kris-2)

The task Molly chose was to prove that $2^n \geq n$ *for all n*, by induction or in any other way. Usually in the framework of school mathematical curriculum students learn proofs by induction without formal learning of the Peano Axioms. Molly's solution included the use of this axiom. She provided a precise solution of the task (that we do not display herein) and then wrote:

> Peano's axiom (In each subset of natural numbers there is a minimal element) serves as the basic assumption for the set of Natural numbers. The other one is the axiom of induction. This topic belongs to the Number Theory. The use of Peano's axiom makes solutions shorter by many times and makes solutions possible at all. (Molly-3)

In these three examples we identify three different *ways* in which AMK can be implemented: Rachel described a situation of *creating a task* for her students, where she applied her knowledge related to matrices and determinants acquired in a course of Linear Algebra. Similarly, Kris chose compatible numbers when creating a task for his students, building on the knowledge of Pythagorean triples acquired in a Number Theory course. Pat mentioned a teaching situation in which she was able to *check students' solution* rather "fast" using her knowledge of Calculus. Molly's example included a specific task, for which she was able to *produce a proof* using her AMK of Number Theory, in addition to the "standard" proof expected in school.

Whereas our request, both in the interviews and in the written questionnaire, invited respondents to draw connections between their AMK and teaching or curriculum, in many cases it either received no attention or was misinterpreted in two different ways: teachers demonstrated AMK without relation to teaching or school curriculum or provided curriculum-based related examples that do not require AMK.

Examples of AMK Beyond School Curriculum

Searching for tasks that require AMK or are related to AMK, five teachers provided examples of tasks that are out of the scope of the secondary mathematical curriculum in its most advanced stream. For example, Ken's task was "Find $\int xe^x dx$." His solution included integration by parts which exemplifies his AMK but does not attend to the request to provide examples related to teaching situations from personal experience or tasks related to the school curriculum.

Dina's example also relied on content beyond the school curriculum:

> Given a sequence of numbers $a_n = \frac{5n-3}{2n+1}$, prove that for this sequence $\frac{2}{3} \leq a_n \leq 2\frac{1}{2}$ for any n. (Dina-1)

In the proof provided in her written work she relied on the calculation of a limit, a notion that is not explored in secondary school curriculum. As in the example provided by Ken, her choice demonstrated her AMK but did not attend to teaching or curriculum.

Simon exemplified his AMK by discussing cardinality of infinite sets, the theorem that establishes one-to-one correspondence between rational and natural numbers, and Cantor's theorem that proves that cardinality of real numbers is higher than the cardinality of rational numbers. Although this knowledge is extremely helpful when introducing students to irrational numbers, Simon did not make an explicit connection to the curriculum content or to possible students' questions about the "size" of different infinite sets. (Simon-1)

Examples of Curricular Mathematical Content without AMK

In responding to Question 4, Daniel suggested the following task:

> Given two points A(7,5) and B(3,1). Write the equation of a circle with diameter AB. (Daniel-2)

In responding to Question 3, Terry wrote the following:

> When doing inverse functions, the students will have difficulties in applying the algorithm to trig functions or exponential functions. The connection between log and exp functions is based on the property of inverse functions ($f \circ f^{-1}(x) = x$). Some of the log properties that students need to know are based on this connection (i.e., $b^{\log_b x} = x$). (Terry-2)

Both excerpts relate to topics of secondary school curriculum and are usually not explored further in undergraduate mathematics courses. In a classroom conversation with peers Daniel noted that these tasks were difficult for his students and thus were considered as related to AMK. Terry acknowledged her students' difficulty when presenting her example. We note that while these tasks may present some difficulty, they are not beyond the reach of a student who learned these topics within the school curriculum.

In fact, examples of tasks or topics in which students experience difficulty were provided by six teachers. This could be a possible misinterpretation of the request in the questionnaire or lack of ability to connect personal AMK to teaching practice.

Meta-Mathematical Issues

As mentioned earlier, by “meta-mathematical” issues we refer to cross-cutting themes that may appear within any mathematical content. The following are illustrative examples of the identified issues.

Proof. Paul noted in his interview that he finally understood the meaning of mathematical proof after failing a first course in analysis. He claimed this made a profound impact on how he teaches “proof,” but he was not able to articulate this claim with any examples. (Paul-interview)

Language. Nadia stated that undergraduate mathematics made her very sensitive to mathematical language, and this influences her teaching in not allowing students to use sloppy expressions. As an example, she shared a recent exchange in which a student said, “these angles make 180” and she asked him to rephrase, aiming for an expression like “the sum of the measures of these angles is 180 degrees.” (Nadia-1)

Precision and aesthetics. In the excerpt below Dina articulated further her sensitivity to mathematical language and brought forward the notion of mathematical beauty:

> The importance of mathematical discourse connected in my mind to my studies in the university. I pay attention to the preciseness of mathematical language used in my classroom and explain to my students differences in the precise and imprecise mathematical formulations. I also am aware of the aesthetics that exists in mathematics and try to bring to my classroom examples of beautiful solutions and encourage students finding beautiful solutions. (Dina-2)

Connection to History

Pat noted that she learned in a university that logarithms were invented independently from the exponential function. As such, while the local curriculum introduces logarithms as the “inverse” of exponential notation, she introduces logarithms consistent with their historical development, building a relation between geometric and arithmetic sequences (Pat-2). Greg noted that he learned in a university course about the Pythagoreans and their decision to keep secret their discovery of irrational numbers. He often uses this story to motivate students when he teaches the topic of irrational numbers (Greg-1).

We note that although all the above situations exemplify pedagogical content knowledge and describe valuable teaching moves or approaches, they do not really rely on advanced mathematical content.

Purposes and Advantages of AMK

As mentioned, the mathematical examples that teachers provided were rather limited. When asked to provide specific examples of situations or problems related to AMK (Questions 3 and 4), most teachers responded with general claims about teaching and curriculum. The most prominent theme, that at times is implicit, is identified in the following excerpt:

> When I think of the relevance of my AMK in relation to my teaching, I do see that my university math is helpful for my math teaching, not necessarily in terms of the specific math content I learned, but more in terms of the learning of how to handle math. The more math I learn and do, the more ways I know of handling math, which I take into my math teaching. (Zack-1)

This idea of "not necessarily specific content" is the main overarching theme to which participants adhere. However, if what is perceived as useful or essential is not a mathematical content, then what is it? For Lisa, in the interview excerpt presented above, it is a way of thinking. Steve answers this question in the following quote:

> The textbook usually will cover what is relevant to the curriculum but it is the extra knowledge that one has to allow for "good" insight to teach. (Steve-1)

What Steve and others consider as "good" insight to teach, and what Lisa refers to as "a way of thinking" are elaborated below by identifying several repeating and overlapping themes in teachers' responses.

Connections within and Beyond the Curriculum

The issue of connections was one of the main themes identified in teachers' responses.

> Understanding the connections and the relationships between the topics of the different curricula is very important and, I think, essential to a teacher of mathematics. Without AMK I would not be as equipped to make the connections or to discuss different questions in the classroom. (Meg-1)

These teachers' views were in line with NCTM (2000) standards that stressed the importance of connections. According to the Standards, without connections students have to rely on their memory only and to remember many isolated concepts and procedures. To connect mathematical ideas means linking new ideas to related ideas considered previously and solving challenging mathematical tasks by thinking how familiar concepts and procedures may help in the new situations.

While the importance of connections was acknowledged, what specific connections teachers had in mind often remained unclear. However, in teachers' responses that mentioned connections we identified three ideas: connecting school mathematics to more advanced mathematics, connection between different mathematical topics, and connection to "real life."

Seeing the whole picture: connecting within curriculum and to more advanced topics. Teachers suggested that AMK leads to the ability to see a "better picture" or a "whole picture" of the subject. Adam referred to this metaphorically as a "sense of terrain." Max and Kris make a more explicit connection to a "subsequent mathematics course" and "college mathematics." Steve also mentions mathematics beyond the scope of school curriculum, but he also attends to the "bridge" among different curriculum topics.

> AMK gives me a better picture or view of where my teaching is headed. It is a sense of terrain. You have to have a sense of building blocks. Most of the work is cultivated here in high school. (Adam, interview)

> If I perceive that a high school topic is going to have some application down the road in a subsequent mathematics course, I can make the decision to play up that topic, and downplay some other "less critical" aspect of the curriculum. (Max-1)
>
> Teachers with advance mathematics knowledge (AMK) are more likely to present problems in contexts that are familiar to the students and to make a connection between the school mathematics and college mathematics because they know what mathematics courses students will have next and are intimately familiar with what is important for them to know before they get there. (Kris-3)
>
> Textbooks quite often do not bridge any divides between topics. The textbooks discrete presentations are left to the students and/or the teacher to interpret. AMK can guide a teacher to understand and present connections. AMK provides an enrichment of the material. It elaborates the high school curriculum, proves or gives basis to high school curriculum, and, more prominently, it elaborates mathematics beyond the scope of the high school curriculum. (Steve-2)

Considering alternatives: connecting different topics or solutions. Teachers claimed that AMK was needed to consider alternatives, such as alternative solutions, alternative strategies, and alternative methods of explaining or presenting material for their students. As multiple solution tasks are perceived as synonyms of connecting tasks (Leikin & Levav-Waynberg, 2008) this direction in teachers' talk serves another indication of teachers' ability to see in AMK a tool for promoting mathematical connections.

> If a student does not understand the way I am explaining a topic my AMK gives me a little more breadth of topics to relate the current one to and I can think around the topic more to find other methods of explaining it. Many teachers who don't have a Math background rely heavily on the text they are given and find it more difficult to introduce alternative solutions or methods to problem solve. When teachers from other curricular areas teach Math there is sometimes misinformation as a result of not remembering the "right" way to do the question or a rigid approach to there being only one way to get to the solution which can limit students' understanding by exposure to multiple types of solutions. (Selina-2)
>
> AMK helps teachers understand multiple representations and abstraction of mathematics concepts and are able to use these representations and abstraction to further students' understanding. (Kris-4)
>
> AMK definitely supports the ability to learn and teach more than one strategy to perform a particular mathematical procedure. By knowing different approaches one can be aware of what students may suggest and where their solution may originate. (Steve-3)

Real life and society: connecting to mathematics beyond the classroom. A few teachers saw the major importance of their AMK in their ability to relate school mathematics to mathematical applications in society. At times this was perceived essential in responding to students' questioning the usefulness of what was learned.

> What I do use is my understanding of how math is really used (and it's limitations) to model real life problems such as structural design, and how probability is actually used, in designing to withstand a 50-year flood, for example. (Daniel-3)
>
> I think a math teacher's toughest job is in public relations. That is, how to convince students that math is not only important to them but to society. Without this argument, there is a lack of motivation for the students. AMK has helped me in knowing and then passing along the areas in which math is important at a higher level. Examples include, development of technology, game theory, meteorology etc. (Greg-2)

> I often use AMK to show students possible real life applications of mathematic. For example, when teaching factoring, I told students that internet security encryption is so hard to crack because it is very hard to find the factors of the encryption. (Zack-2)

However, Maya presented a rather critical view on the focus on real life applications:

> Sometimes, I think that because we often focus our energy on making things relevant and "real-life" for the students, we miss out on opportunities to demonstrate and explore the beauty of mathematics. (Maya-2)

Comfort and Confidence

Teachers' confidence and comfort are central characteristics associated with their autonomy (Krainer, 2001). AMK provided teachers with both mathematical and pedagogical tools that allowed them to cope with complexity of the profession. As seen in the quotes below, knowing more than students know, and more than students need to know, provides teachers with comfort and confidence.

> I often use AMK in planning the structure of a lesson as I try to introduce a new topic through a problem or situation. The courses I took allowed me the greater confidence to look at real-life situations and pose problems of the "what if" type. I believe that the further your knowledge extends beyond the material you teach then the more comfortable you are in doing this. (Ken-2)
>
> The mathematical knowledge acquired through the university study, fostered a lifelong love of teaching mathematics and established a comfort with numbers, theories, and algorithms. Thus the advanced mathematical knowledge has enabled me to transmit knowledge to students with clear and confident manner. (Sandy-2)

As mentioned in the above excerpts, comfort and confidence with mathematics, as provided with acquiring of AMK, serves teachers in planning for teaching, in posing problems for students and in delivering specific content. Among additional issues, related to comfort and confidence with the subject matter, teachers mentioned their ability to perform tasks quickly and to address students' questions related to the topics taught as well as questions related to future careers.

Doing things quickly. The theme of doing things "quickly" which is closely related to comfort and confidence appeared in several variations. On one hand, as David pointed out, his experience allows him to to do school mathemamatics faster. On the other hand, AMK allows teachers to use methods that are not expected from students, such as Matrix Algebra (Maya), Vector Algebra (Simon), and Calculus (Max, Maria), which leads them to the solution faster than using the methods that students learn to use. Excerpts from the responses below exemplify the advantages of speed.

> Just having further years of doing proofs, calculations and algebra just makes me more able to do it quickly and see mistakes in student work. (David-2)
>
> AMK provides teachers with techniques to check some answers more quickly. For example given the coordinate of three vertices of a triangle ABC, students should do many things to find the area of the triangle, while the teacher can check immediately by vector product of $AB \times AC$. (Simon-2)
>
> Using matrix algebra to more quickly determine if a student's solution to a system of equations is completely correct. (Maya-3)

> When we are studying third and fourth degree functions I will sometimes use the first derivative to find points of inflection, rather than push the buttons on the graphing calculator. I do this only because it is often faster to take the derivative than it is to use the calculator. Of course, I do not do the calculus with students: only when I am figuring a problem out on my own. (Max-2)
>
> Optimization problems—These problems can be solved more quickly using calculus even if that it not the intent. I have used calculus to get the answer to a problem quickly while the students are solving the problem without the use of calculus! (Maria-1)

Responding to students' questions. About half of the teachers mentioned that AMK was needed to address students' questions. These questions were of different kinds: requiring extension or clarifications of the material, and those wondering about the usefulness of what is being learned.

> Math teachers need AMK to have that comfort level with students' questions. (Steve-4)
>
> AMK is needed to deal with students' questions: why can't you divide by zero? What is infinity? Is it real? What is an imaginary number? etc. (Meg-2)
>
> The most important qualification for a math teacher, in my opinion, is to enjoy math and to communicate this joy to the students. A crucial test is how a teacher responds to the student's question "Why are we learning this?" (Ron-1)
>
> It helps me to answer the persistent questions "When am I ever going to use this?" and "What is this good for?" (Darlene-2)

Looking to the future. Furthermore, teachers' confidence, which is based on their use of AMK, is rooted in their ability to provide students with advice about their future career.

> At the senior level I often have students ask me about post-secondary mathematics. This is more of a "counselling" situation than it is mathematical; however, I can suggest to a student that he/she either enrol or not depending on his/her interests, abilities, and goals. And I can make that recommendation based on what I know about the student, and what I know about post-secondary mathematics. (Michael-1)
>
> I usually introduce the topics that I listed in Q2 by telling them what they will lead to. Example, when I introduce systems of equations in grade 11, I will let them know that they will eventually need this knowledge to help them to solve simple econ and gaming problems using matrices. (Terry-3)

Ways of Thinking: Attending to Interrelationship

We acknowledged earlier that the ways and purposes or advantages of AMK usage are interrelated, and that the themes identified in the previous two sections are intertwined with each other. It appears that the themes that tie the two categories together are those of "ways of thinking," "depth or insight," and "problem solving."

> Teaching math forced me to think about each topic in depth, a task that would have been difficult without having AMK. (Hanna-2)
>
> In teaching prime factorization AMK gave me a much greater insight to the nuances of primes, their uses, and interesting ways to help bring the topic to life. AMK gave me many more ideas and a greater depth of knowledge regarding basic number theory and, I believe, a more enthusiastic way of teaching the subject. [. . .] I believe AMK has given me the ability to both ask and answer the question "why is that?" (Ken-3)

> What it (AMK) did was it made me continue to like math, to be amazed by it and enjoy it. It is not the technical stuff that we did. It is exploring and discovering. It is more the problem solving stuff and working through things, and the joy you get from doing that, that has connected now to my teaching. (Brian, interview)

Furthermore, not mentioned in a separate section but appearing across various themes are issues of depth of understanding (e.g., Hanna-1, Hanna-2, Ken-3), providing students with motivation (e.g., Greg-2), ability to engage students (e.g., Maya-1) and to enrich their experience (e.g. David-1, Sandy-1, Selina-1, Steve-2), problem solving ability (e.g., Ken-2, Hanna-1, Kris-2, Selina-2), joy of mathematical experience (e.g., Maya-1, Ron-1, Brian-interview), work ethics and persistence (e.g., Lisa-interview), love of teaching mathematics (e.g., Sandy-2), and quality and rigor of mathematical discourse (Dina-2).

SUMMARY AND DISCUSSION

It is generally agreed that secondary school mathematics teachers must have good preparation in their subject matter. However, the interpretation of "good" in this context varies significantly among different teacher education programs, from requiring just a few specific university-level mathematics courses to a full Bachelor's degree in mathematics, as a subject matter competency condition for teacher certification at the secondary level. While undergraduate content requirements for secondary teachers exist almost everywhere, it has not been investigated *how* mathematical knowledge acquired at the undergraduate level—referred to here as AMK, "advanced mathematical knowledge"—is manifested in teaching practice. Our study initiates a conversation on this issue by describing secondary teachers' perceptions of when, how, what for, and to what extent their AMK is used in their teaching practice.

While teachers' knowledge received wide attention in mathematics education research, the main focus has been on having knowledge rather than on using this knowledge in teaching practice. However, more recent studies attended not only to what teachers know or need to know, but also to using mathematics in teaching (Adler & Ball, 2009). Our main contribution is in extending a conversation about the usage of subject matter knowledge, focusing on AMK, and attending to teachers' perception, as a step toward studying their practice. While no generalizability can be claimed by considering personal descriptions of teachers, our study provides a foundation for future research that looks at teachers "in action" and also attends to the usage of subject matter knowledge beyond what is taught by other professionals teaching mathematics: mathematicians teaching undergraduate courses and elementary school teachers.

Returning to specific research questions posed in this study, we demonstrated that the extent to which teachers believe they use their AMK varied significantly, from claims of using it "not at all" or "rarely" to using it "all the time." Further, while all teachers mentioned several mathematical topics where AMK was helpful, there was a significant difficulty in generating specific problems or recalling particular situations of AMK usage. Although some content-based examples referred, as requested, to secondary school curriculum, other examples focused on "harder" tasks within the curriculum or extended beyond typical school expectations. However, the majority of teachers' examples of their AMK usage related to meta-mathematical issues (proof, elegance of solution, rigor of language) or to pedagogical issues. Even when the teachers presented mathematical examples they included considerations for teaching that explained their choices.

When the teachers illustrate how they use AMK in their teaching, in addition to the categories presented in the analysis, we can differentiate between teacher-self-oriented and student-oriented usage of AMK. Teacher-self-oriented usage is revealed when the teachers think how AMK supports their mathematical expertise and skills, such as problem solving or "ways of thinking." Student-oriented usage of AMK is aimed at developing students' understanding, providing alternative explanations or multiple solutions to help students and to make the lesson more interesting, broadening and enriching students' knowledge beyond the curriculum, supporting students' motivation, and encouraging their curiosity. Student-oriented use of AMK is also present in teachers' responses to students' questions, whether these questions concern specific mathematical topics, "usefulness" of particular contents, or consideration of their future careers. Teachers' confidence was one of the central factors evoked when the teachers considered their usage of AMK in teaching. This issue connects self-oriented usage—personal confidence with the subject matter, to student-oriented usage—confidence in teaching and in attending to students' needs.

As mentioned previously, despite our request to provide specific mathematical examples, the majority of participants identified the usefulness of AMK in general terms related to teaching and pedagogy. The gap between university mathematics and mathematics taught in secondary school is further evidenced by the difficulty that many teachers experience when asked to articulate specific examples of using their AMK. We wonder whether this gap is inevitable or is it a result of the curricula implemented at both university and high school levels. We question, and investigate in a further study currently in progress, whether indeed mathematical-content usage of AMK is rare or whether teachers are simply unaware of such usage and, as such, they are having difficulty in articulating it. Inviting teachers to respond to specific prompts or classroom scenarios, rather than general questions on AMK usage implemented in this study, as well as long-term classroom observations, are necessary in addressing this question. While some teachers interpreted the fact that specific examples were not readily available as uselessness or irrelevance of AMK, Ron put this in very optimistic terms:

> So was my university math useless as far as teaching high school math is concerned? Not at all! My familiarity with math, my enjoyment of math, and my confidence in math are far more important for teaching than knowledge of any specific AMK topics. And probably I could only have achieved this fluency by means of taking a number of university courses in mathematics. (Ron-2)

Although we agree with Ron about the importance of enjoyment of mathematics and confidence, we do not wish to disregard the significance of content-related mathematical knowledge.

Having acknowledged a variety of pedagogical and meta-mathematical issues, our study calls for identifying explicit content-based connections between AMK and mathematics taught in secondary school. An explicit awareness of these connections and an extended repertoire of examples will inform the instructional design in teacher education.

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