When Teaching Becomes Learning

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This research examines the role of teachers' content knowledge during the implementation of mathematics education reform. Current mathematics education reform efforts require teachers to learn in the act of teaching. I claim that this learning occurs as teachers negotiate among 3 areas of their content knowledge: their understanding of the subject matter, view of the curriculum materials, and knowledge of student learning. The data for this study come from observations and videotapes of 2 teachers implementing a reform-based linear-functions unit in a high school algebra class. The focus of the article is a detailed analysis of 1 lesson that illustrates the process through which these negotiations occur and the learning that takes place as a result.

Reform in mathematics education places numerous demands on teachers. They are asked to introduce new technologies, engage students in meaningful activities, and create communities in which students can discuss and reflect on their learning (National Council of Teachers of Mathematics [NCTM], 1991, 2000). At the forefront of this challenge are questions concerning the knowledge that teachers bring to their work and how this knowledge must develop if teachers are to manage the complex demands of reform.

This article examines the role of *content knowledge* as teachers engage in the types of instructional practices recommended by current mathematics education reform. By content knowledge, I refer to the body of knowledge comprising subject matter knowledge and pedagogical content knowledge (Shulman, 1986). In brief, *subject matter knowledge* is an understanding of the facts and concepts within a domain. *Pedagogical content knowledge* is knowledge specifically for teaching the domain, including understanding how to present the facts and concepts to facilitate learning and knowledge of the typical understandings and misunderstandings of students. Thus, I investigated the ways in which teachers' under-

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standings of mathematics and mathematics teaching affect their implementation of mathematics education reform.

The central claim of this article is that, because of the particular nature of current mathematics reform efforts, implementing this reform requires that teachers learn in the act of teaching. This claim is not the more general one that implementing any novel instructional program requires a teacher to learn; rather, I argue more specifically that mathematics education reform—as it is currently realized—requires teachers to learn as they teach. In particular, the following two types of learning are involved.

1. *Existing content knowledge is modified during instruction.* Experienced teachers have a wealth of established routines for thinking about and teaching particular subject matter; in fact, this is the basis for much of their expertise (Leinhardt & Greeno, 1986). However, the current reform requires that teachers adapt and modify these familiar approaches. Thus, rather than using established practices, teachers must apply their existing content knowledge more flexibly.
2. *New content knowledge is developed during instruction.* Implementing reform also promotes the development of new pedagogical routines and new understandings of the domain for the teacher. I argue that this learning occurs as teachers negotiate among three areas of their content knowledge: their understanding of the domain, view of the curriculum materials, and knowledge of student learning.

In making this claim, this article addresses three related issues. First, I discuss the role of content knowledge in teaching and describe the close ties that form between teachers' subject matter knowledge and their pedagogical content knowledge. Second, I look at the role of content knowledge during the implementation of reform. In particular, I introduce a framework that highlights the teacher learning that occurs in this context. Third, I describe the process through which this learning occurs—how teachers construct content knowledge during instruction, in the context of reform.

The article focuses on a lesson from a high school algebra class in which the teacher uses a new reform-based linear-functions unit. During the lesson, the teacher negotiates among several aspects of her content knowledge, and she learns as a result of this negotiation process. The lesson thus serves as an example of the ways in which teaching can become learning for the teacher. Before turning to the example, I briefly review the literature on teachers' content knowledge and describe the context in which this study was undertaken.

TEACHERS' CONTENT KNOWLEDGE

Until recently, research on teachers' subject matter knowledge was largely absent from the literature on teaching. It came to the forefront when Shulman (1986) claimed that teachers have *pedagogical content knowledge*—subject matter knowledge that is specialized for teaching. Shulman argued that, in addition to understanding the relevant facts and concepts in a domain, teachers need to understand how to teach a particular topic. In this article I consider the interaction among teachers' subject matter knowledge and two components of their pedagogical content knowledge: knowledge of students' understanding and knowledge of curriculum.

Current models of teachers' knowledge contend that both knowledge of the content (viz., subject matter knowledge) and knowledge about how to teach that content (pedagogical content knowledge) are critical for effective teaching (Ball, 1991; Fennema & Franke, 1992; Shulman, 1987). However, such models generally fail to explain how this knowledge is used in the act of teaching. This research illustrates this relation by explaining the role of teachers' content knowledge in a particular context: during the implementation of reform. To do this, I first elaborate on what it means for teachers to implement mathematics education reform.

What Is Reform Teaching in Mathematics?

Up to this point, I have not been precise about what is meant by "the types of instructional practices recommended by reform." Part of the problem, of course, is that there is no one definition or model of this practice. Instead, the forms of instruction called for by current mathematics education reform are often characterized in different ways by key phrases such as "teaching for understanding," "building a community of inquiry," or "mathematics for all." In addition, understanding the nature of this reform has evolved, as exhibited in the creation of revised national standards (NCTM, 2000).

Despite this variation, there is consensus on some key issues—"reform mathematics teaching," whatever the details, requires that teachers change what they teach and how they teach it. The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional Standards for Teaching Mathematics* (NCTM, 1991), and *Principles and Standards for School Mathematics* (NCTM, 2000) help to provide a vision of these changes by describing practices considered to be in line with mathematics education reform and by presenting assorted vignettes of such practice. For this article, I have chosen three aspects of reform-oriented practice to discuss in further detail. All three are commonly agreed to be important facets of implementing mathematics education reform. In addition, they draw heavily on teachers' content knowledge and are therefore of particular importance for this study.

Changes in instructional materials. One important aspect of engaging in mathematics reform is that teachers must change the content of instruction. Although the emphasis of instruction has traditionally been on learning mathematical procedures, today teachers are asked to focus on mathematical concepts, multiple representations of those concepts, and connections among them. In many cases, teachers are expected to use new curricula that have been designed with these goals in mind. Yet research has shown that such curriculum materials are not always effective agents of change. Although in some cases teachers do adapt new materials successfully, in other cases teachers transform these materials to be used with their familiar instructional routines. As a result, teachers who use reform-based curricula do not always appear to be implementing reform in the ways intended (Cohen, 1990; Putnam, 1992; Sherin, 1996a). There clearly is more to the implementation of reform than simply adopting new instructional materials; a teacher's pedagogical strategies must change as well.

An adaptive style of teaching. A second key aspect of implementing reform is that teachers are asked to use new instructional strategies; in other words, teachers must not only use new materials, but they also must use these materials in new ways. For example, in traditional mathematics instruction, teachers plan relatively detailed lessons in advance and then attempt to carry out these plans during instruction. In contrast, current mathematics reform requires a more adaptive style of teaching in which teachers attend to the ideas that students raise in class. Consider, for instance, a typical lesson in which students work on a problem and then explain their solution strategies to the class. The students' solutions cannot be predicted entirely in advance; thus, the teacher must listen to the ideas that students raise in class and then use that information to decide how to proceed.

This adaptive style of teaching has been described by researchers in a number of different ways, including *inquiry* (Ball, 1993), *discovery* (Hammer, 1997), and *improvisation* (Heaton, 2000). In all of these views, the teacher must make on-the-spot decisions concerning what mathematics to pursue and how to pursue it. As Chazan and Ball (1999, p. 7) explained, "teacher moves are selected and invented in response to the situation at hand, to the particulars of the child, and to the needs of the mathematics."

A focus on classroom discourse. A third key component of implementing mathematics reform concerns the teacher's role in directing classroom discourse. In contrast to the once-popular "teaching by telling" paradigm, orchestrating classroom discourse is now seen as a central model for instructional practice. The *Professional Standards for Teaching Mathematics* (NCTM, 1991) recommended that teachers coordinate this discourse by posing appropriate questions for students to consider, listening to students' ideas, and helping students to explain and justify their ideas in class. In addition to using discourse as an opportunity to draw out and analyze students' thinking, classroom discussions are also a time for teachers to insert mathematical ideas and explanations where appropriate. Chazan and Ball (1999) provided examples from their own teaching in which they made substantive mathematical comments during a lesson. They attempted to respond to students' ideas that came up during class and to provide a mathematical perspective that could help students further pursue their ideas.

These three aspects of instructional practice—adopting new materials, using an adaptive style of teaching, and directing classroom discourse—elaborate on what it means for teachers to engage in the types of instruction called for by current reform efforts. With this in mind, I turn now to the role of teachers' content knowledge in implementing these forms of instruction.

Content Knowledge in the Context of Reform

Much research on the implementation of mathematics education reform focuses on the difficulties teachers encounter as they attempt to apply reform measures. A number of studies have claimed that these difficulties are due, in part, to teachers' content knowledge. Such research has found that either teachers do not have enough content knowledge or that what they do know is not the "right" content knowledge.

Research has found that teachers implement mathematics reform through the lens of their current practices (Cohen & Ball, 1990). However, rather than reinterpreting familiar instructional strategies in terms of reform recommendations, veteran teachers may simply add new practices on top of existing classroom structures (Cohen, 1990). For example, a teacher may have students work with manipulatives, but only to complete rote computations (Peterson, 1990). Or a teacher may develop a lesson on a new mathematical topic, but ask only closed-ended questions during the lesson. In such cases the teachers' existing content knowledge constrains their interpretation of reform recommendations and limits their ability to make changes in their practices.

In other cases, researchers have documented that teachers do not have the depth of content knowledge required to implement reform effectively. Thus, rather than having to "unlearn" a set of familiar teaching practices, some teachers simply need to develop new understandings of the domain. For example, consider Putnam's (1992) examination of a fifth-grade teacher's efforts to introduce the topic of averages into the curriculum. Without a deep understanding of the mathematics involved, the teacher was unable to judge the appropriateness of using the averaging procedure presented in the textbook for problems that she had developed on her own. As a result, the teacher and class extended the procedure incorrectly to other types of problems. Similarly, Borko et al. (1992) observed a preservice teacher conducting a lesson on division of fractions. At one point, a student asked why one was supposed to invert and multiply. In response, the teacher wanted to provide a conceptual explanation to demonstrate the meaning of the procedure. Nevertheless, her own understanding of the domain was lacking, and she was unable to come up with an appropriate example at that moment.

As these examples show, reform calls for changes in the ways that teachers help students learn as well as changes in teachers' own understandings of mathematics. Yet such research is limited in that it focuses on the ways in which content knowledge constrains teachers' ability to implement reform. In contrast, in this article I attempt to characterize how content knowledge is used in situations in which teachers do successfully implement reform.

Content knowledge complexes. Although the focus of this research is on the relation between content knowledge and the implementation of reform, consider for a moment how content knowledge is applied in familiar teaching situations. As with human cognition in general, teachers' actions during familiar teaching situations depend on their prior experiences in similar contexts. In particular, I claim that previous teaching experiences result in strong connections between the content of a lesson and the instructional strategies the teacher has used to teach that content. In other words, pieces of subject matter knowledge and pedagogical content knowledge that are accessed together repeatedly during instruction become connected. I call these connected bodies of knowledge *content knowledge complexes*. Furthermore, I claim that, during instruction, teachers access these content knowledge complexes rather than drawing on individual pieces of subject matter knowledge and pedagogical content knowledge.

When teachers consider a particular topic, they do not think just in terms of their subject matter knowledge or their pedagogical content knowledge; instead, they tend to call on both types of knowledge. As a result, teachers almost automatically apply the pedagogical routines associated with a particular piece of content. For example, if a teacher has taught linear functions many times before, when it is once again time for the lesson on slope, the teacher will draw on his or her familiar pedagogical strategies to teach the lesson.

To understand how content knowledge complexes relate to other descriptions of teacher knowledge that appear in the literature, it will be helpful to distinguish between claims about the *form* of teacher knowledge and claims about the *content* of teacher knowledge (Schoenfeld, 1998; Sherin, Sherin, & Madanes, 2000). In brief, claims about the form of teacher knowledge concern the structure of the knowledge that teachers possess. For example, *curriculum scripts* (Putnam, 1987), *agendas* (Leinhardt & Greeno, 1986), and *lesson-images* (Morine-Dershimer, 1978–1979) are all constructs that attempt to describe how teachers' knowledge is structured and organized. In contrast, claims about the content of teacher knowledge concern what the knowledge is about. For instance, in presenting the idea that teachers have pedagogical content knowledge, Shulman (1986) made claims about the content of teachers' knowledge but not about the structure of this knowledge.

With the term *content knowledge complex* I primarily address the content of teacher knowledge. Specifically, I claim that there are larger elements of teacher knowledge that cannot be categorized either as subject matter knowledge or as pedagogical content knowledge. With respect to form, content knowledge complexes offer a weaker claim: simply that there are composite structures within teachers' knowledge, with tight interconnections among the components. Although this claim is relatively weak, it does capture some of the phenomena that have been described by researchers who primarily made claims about the form of teacher knowledge. For example, a number of researchers discuss the automaticity that allows expert teachers to manage the complexity of teaching by describing teachers' use of routines (Leinhardt & Greeno, 1986; Schoenfeld, 1998) and curriculum scripts (Leinhardt, Putnam, Stein, & Baxter, 1991; Putnam, 1987). This automaticity is also captured, to a certain extent, by my claim that there are tightly integrated structures containing both subject matter knowledge and pedagogical content knowledge. Because of the strong connections between subject matter knowledge and pedagogical content knowledge, teachers almost automatically implement the pedagogical routines associated with the content under consideration.

Role of content knowledge complexes in novel teaching situations. Thus far, I have discussed the role of teachers' knowledge in familiar teaching situations, and yet the implementation of mathematics education reform requires an adaptive style of teaching in which teachers will find themselves, at least part of the time, in novel teaching contexts.

As can be expected, applying their existing content knowledge complexes does not always enable teachers to meet the demands of novel teaching situations. Because a content knowledge complex specifies the pedagogy for teaching a particular topic, relying exclusively on these knowledge structures may limit a teacher's options and does not promote the responsiveness called for by reform. Thus, in cases in which teachers are trying to use a more adaptive style of teaching, they need to develop other ways to apply their content knowledge.

I claim that as teachers engage in the types of instruction called for by current reform, they do in fact develop methods of applying their content knowledge that support these practices. I have found that teachers, rather than relying on their content knowledge complexes, develop new content knowledge by engaging in a cycle of negotiations among their understanding of the lesson, views of student learning, and knowledge of mathematics. The goal of this article is to illustrate how such negotiations occur and to demonstrate the ways in which this learning supports teachers' efforts to implement reform.

The Teaching and Learning of Linear Functions

In describing the ways in which teaching becomes learning for the teacher, I present a detailed analysis of the implementation of one lesson from a reform-based linear-functions unit. As described earlier, mathematics education reform has called for changes in the ways that numerous topics are taught, including the concept of functions. No longer is computational fluency the single focus of what students are expected to learn about functions and their graphs. Instead, as described in the *Principles and Standards for School Mathematics* (NCTM, 2000), students are expected to analyze functions both quantitatively and qualitatively, with conceptual understanding as a key goal for students' learning. Furthermore, the *Principles and Standards for School Mathematics* explains that knowledge of functions is a central component of learning about algebra and a topic that should be emphasized throughout students' school years.

Despite the importance of functions, previous research has shown that they can be quite difficult for both students and teachers to understand (Even, 1993; Leinhardt, Zaslavsky, & Stein, 1990; Stein, Baxter, & Leinhardt, 1990; M. R. Wilson, 1994). One of the challenges is that understanding functions requires one to integrate a number of different mathematical ideas and representations. To be clear, this article does not specifically contribute to the literature on the teaching and learning of functions. However, because this topic is the subject matter of the teaching case that is presented, an understanding of research in this area will be help readers interpret the case. Therefore, rather than synthesizing relevant literature on functions here, I mention particular results while presenting the analysis. In doing so, I draw on research pertaining to proportional reasoning (e.g., Noelting, 1980), ratio-as-measure (Simon & Blume, 1994), and the process and object conceptions of functions (e.g., Sfard, 1992).

RESEARCH DESIGN

In this study I investigate the role of content knowledge as teachers use a novel linear-functions curriculum. Although the teachers have taught linear functions before, the new curriculum is quite different from their previous materials. Instead of focusing on the procedures used to manipulate linear equations, the new unit emphasizes connections among representations, conceptual understanding of slope and intercept, and real-world contexts for investigating linear functions. Furthermore, the instructional practices described in the materials are designed to provide opportunities for teachers to elicit students' thinking and to use these ideas as the basis for exploring mathematics. Thus, examining the implementation of this unit provides an opportunity to study teachers' attempts to engage in the types of practices recommended by current reform efforts.

A Research-Based Linear-Functions Curriculum

For more than 10 years, the Functions Group at the University of California, Berkeley, has explored the teaching and learning of linear functions. The group has examined the complex connections among algebraic and graphical representations of linear functions (Moschkovich, Schoenfeld, & Arcavi, 1993), students' conceptions of functions and graphs (Moschkovich, 1992; Schoenfeld, Smith, & Arcavi, 1993), and teaching and tutoring in this domain (Magidson, 1992; Schoenfeld et al., 1992). On the basis of this research, a 6-week unit on linear functions was developed (Lobato, Gamoran, & Magidson, 1993).

As part of a collaboration between the Functions Group and a local public high school, two experienced teachers volunteered to implement the new unit in their algebra classes. Unlike the 2 to 3 weeks usually given to address this topic, the new unit is designed to last 6 weeks. During that time, students explore the use of tables, graphs, and equations to represent linear functions. They examine connections among these representations and judge the appropriateness of using different representations for particular types of problems. Students also work with a number of real-world situations to examine slope and intercept. Particularly relevant for this article is the use of a staircase metaphor to investigate the concept of slope.

In addition to providing activities for students, the unit also includes a set of materials designed to support teachers' use of the activities. A teacher's guide describes each activity in the unit and discusses the intended purpose of each activity. There are additional notes listing possible student solutions as well as confusions that might arise during class, with suggestions for how to proceed. In some cases an appendix explicitly describes how students have used the materials in the past during research with, and pilot testing of, the activities. Support for the teachers was also provided through meetings with members of the Functions Group prior to and during implementation of the unit.

Data

Interviews. Prior to implementation of the unit, I conducted three videotaped interviews with each participating teacher. The purpose of the interviews was to examine the teachers' conceptions of linear functions as a mathematical domain and their ideas about teaching linear functions. In the first interview, the teachers described their previous teaching of linear functions and discussed their ideas about key concepts in the domain. In the second interview, the teachers responded to specific teaching situations presented by the interviewer. For example, in one situation a student approaches the teacher with a novel, and incorrect, method for calculating slope (Kennedy, Ball, McDiarmid, & Schmidt, 1993). For the third interview, the teachers worked together on a series of linear-functions activities that were similar to those in the new curriculum.

Classroom instruction. The new unit was implemented in five Algebra 1 classes from March to May, 1993. Each of the five classes was videotaped daily for the duration of the unit with two cameras to capture classroom interaction. One camera remained fixed on the class as a whole while a second camera in the back of the room followed the teacher. These pictures were merged into one videotape. The teacher wore a wireless lapel microphone so that the audio portion of the tape included whole-class discussions as well as conversations the teacher had with individual or small groups of students.

In addition to the videotape data, I made written observations for more than half of the classes. Building on prior work with the Video Portfolio Project (Frederiksen, Sipusic, Sherin, & Wolfe, 1998), my observations focused on events that provided evidence of the teachers' subject matter knowledge or pedagogical content knowledge. In particular, I noted the teachers' role in discussions of key mathematical concepts, the ways in which the teachers presented curriculum materials, and the teachers' responses to students' questions and ideas.

Video club meetings. During implementation of the unit, the teachers also participated in weekly video club meetings in which the teachers and I watched and discussed excerpts of videotapes of their classrooms (Gamoran, 1994). Our discussions in these meetings focused on the mathematics that was evident in the video. For example, the teachers commented on the concepts being addressed in the lesson, on the representations being used to explore the concept, and on specific student questions or strategies that were shown in the video. We also discussed new instructional strategies the teachers had recently used and how effective they were, which student ideas were novel for the teachers, and why the teachers did or did not enact specific changes in a lesson. The video club meetings were also videotaped.

Analysis

Fine-grained analysis of videotapes was used to examine the teachers in each of the different contexts in which they were studied. The interview and video club data, as well as classroom interactions between the teacher and one or two students, resemble the small-group interactions that are typically the focus of videotape analysis. To extend this methodology to whole-class interactions, I incorporated techniques designed by the Video Portfolio Project, in which video is examined for noteworthy episodes of teaching that are then classified by means of an evolving framework (Frederiksen, 1992). In particular, analysis of classroom interactions focuses on key processes in which teachers' content knowledge is accessed, including the teachers' use of agendas and curriculum scripts, their choice of explanations and representations, and their responses to students' questions (Leinhardt et al., 1991).

Analysis of the data also drew from prior research on student learning in the domain of linear functions. As in the Cognitively Guided Instruction Project (Carpenter, Fennema, Peterson, & Carey, 1988), I extended research on student learning to investigate teachers' pedagogical content knowledge. Thus, prior research on linear functions was used to uncover the teachers' own competence in the domain and to probe how the teachers believed that students learn.

A Framework for Examining Content Knowledge

Through an iterative process of analysis, I identified three classes of interactions between the teachers' content knowledge and the implementation of the novel curriculum (Table 1). Together, these three classes represent the different ways that teachers apply their content knowledge as they attempt to use new materials. In this article I focus on instances that were categorized as cases of *negotiate* because they involve the learning that occurs in the act of teaching that I believe characterizes reform mathematics instruction. However, before turning to those instances I briefly describe each of the three classes (for a detailed discussion of the three different classes, see Sherin, 1996a).

Transform. In *transform,* teachers use their existing content knowledge to implement a new lesson but, in doing so, they instantiate the lesson differently than was intended by the curriculum designers. What happens is that the new curriculum may have features of a lesson that appear familiar to the teacher. In response, the teacher accesses the content knowledge complex that corresponds to those familiar features. Thus, the teacher engages both his or her understanding of the topic and the routines typically used to teach that content. This equips the teacher with a way to teach the novel lesson using known routines, and this is precisely what the teacher does. As a result, the teacher's content knowledge does not change. Instead the teacher changes the lesson to be consistent with what he or she is used to doing.

The notion that teachers may transform new curriculum materials into more traditional-looking lessons is not new. Cohen's (1990) description of Mrs. Oublier provides a classic example in which a teacher's use of new materials is filtered through established routines. Cohen argued, as have other researchers, such as Peterson (1990) and Putnam (1992), that teachers tend to use new materials only to the extent that these materials correspond to their existing practices and beliefs. The case of transform, however, extends such work by describing the mechanism

TABLE 1
Content Knowledge and Teaching a Novel Lesson

Interaction Class	*Content Knowledge*	*Novel Lesson*
Transform	Unchanged	Changed
Adapt	Changed	Unchanged
Negotiate	Changed	Changed

through which this process takes place. In particular, in Sherin (1996a), I explained that as teachers develop expertise, their subject matter knowledge and pedagogical content knowledge become tied together in content knowledge complexes. Although this expertise can be a valuable resource, it also provides a strong set of limitations. Precisely because subject matter knowledge and pedagogical content knowledge are connected, once a content knowledge complex is accessed, teachers' actions are constrained, and they tend to implement established pedagogical routines.

Adapt. In the second case, *adapt,* teachers develop new content knowledge and implement the lesson as planned. An important feature of adapt is that, in most cases, it is novel student comments and actions that trigger teachers to look beyond their content knowledge complexes (Sherin, 1996b). As teachers try to respond to these novel ideas, they develop new content knowledge.

The case of adapt shares an important similarity with other researchers' findings that inquiry into student thinking can provoke changes in teachers' instructional practices (Franke, Fennema, & Carpenter, 1997; Wood, Cobb, & Yackel, 1991). For example, Heaton (2000) claimed that an important component of being able to reform her instruction was learning to hear students' ideas about mathematics as they surfaced in the classroom. Furthermore, teachers' own writings about their classrooms highlight the notion that it is novel student ideas that often prompt teachers to reflect on and rethink their instruction (Schifter, 1996).

To be clear, instances of adapt do involve teacher learning; however, this learning is localized and limited in nature. When faced with a novel student comment, a teacher may invent a new explanation and insert it into the lesson, but this does not lead to a dramatic reworking of the lesson. For that one must turn to the third case, negotiate.

Negotiate. In the case of negotiate, teachers develop new content knowledge and at the same time make changes in a lesson as it unfolds in the classroom. Unlike transform, these changes in the lesson are not a shift to familiar pedagogical routines; instead, the changes involve innovative instructional strategies that are new for the lesson and for the teacher. In other words, the teacher not only develops new content knowledge but also uses this knowledge to interpret the lesson in progress and decide how to proceed. It is the case of negotiate that I believe most clearly illustrates the active learning that is the essence of reform mathematics teaching.

The case of negotiate actually involves a series of changes in teachers' content knowledge and in the novel lesson. I view these changes as a cycle of negotiations among teachers' understanding of the domain, their view of the curriculum, and their knowledge of students' learning. For example, imagine that a novel student idea prompts the teacher to rethink how he or she believes students understand a particular mathematical concept. In turn, the teacher may develop a new way of thinking about that concept and may change the direction of the lesson to address this new thinking. This change in the lesson then provides further opportunities for students to explore new ideas and discuss them with the teacher, which again promotes further developments in the teacher's understanding of the mathematics involved and his or her view of the lesson—and the cycle continues.

In writing about mathematics teacher development Simon (1997) described the "mathematics teaching cycle" as a process through which teachers can learn as they interact with students during instruction. Simon explained that a teacher is "constantly orienting between his or her view of the relevant mathematics and his or her view of the students' mathematics" (p. 76) as well as goals for a particular lesson. Furthermore, Simon claimed that in some instances this inquiry process results in a teacher altering the main thrust of a lesson. These are the kinds of situations I examine in studying the case of negotiate.

The 6 weeks of the novel linear-functions unit consisted of 17 lessons, each lasting 1 to 3 days. I examined each lesson for evidence of transform, adapt, and negotiate, using particular warrants I had identified and tested. The warrants specified a set of observable criteria that would allow an episode to be coded in terms of one of the three cases. With this approach, a single lesson could be coded as an instance of one or more of the cases. Members of the Functions Group verified my coding for over 70% of the instances identified.

The analysis revealed that negotiate occurred the least often of the three cases. Out of the 6 weeks of the unit, transform occurred in 53%[1] of the lessons, adapt in 82% of the lessons, and negotiate in only 29% of the lessons. I believe that the scarcity of negotiate reflects the difficulties teachers encounter when trying to implement reform and indicates the complexity of this form of teaching. More than one half of the lessons involved two or more of the cases. It is interesting that when a lesson involved both transform and negotiate the event that was coded as transform occurred at a different point in the lesson than the event that was coded as negotiate. In contrast, in the lessons coded as both adapt and negotiate the events usually occurred during the same part of the lesson, with adapt generally preceding negotiate.

I now turn to an example of negotiate from the classroom data. I attempt to show the kinds of knowledge on which the teacher drew in using the new curriculum and how that knowledge changed during this process. Because of the detailed nature of the analysis, I am able to present only one such example.

[1]The frequency of transform actually underestimates the teachers' reliance on their content knowledge complexes. In addition to the 53% listed here, there were many situations in which the teachers applied a content knowledge complex but doing so did not change the purpose of a lesson. Therefore, such instances were not coded as transform.

THE REAL-STAIRCASES LESSON

The real-staircases lesson occurred midway through the 6-week linear-functions unit. The lesson was designed to help students explore the features of a staircase that determine steepness. The title of the lesson refers to the fact that students were asked to measure the rise and run of a real staircase located near their homes or school.

In teaching the lesson, the teacher, Lynn Mark (a pseudonym), applied many of the practices recommended by current reform efforts. She used new materials designed to facilitate students' understanding of key concepts, she responded to the lesson as it unfolded in the classroom, and she directed classroom discourse to draw out students' ideas and to suggest additional ideas for students to consider. For these reasons I claim that this lesson serves as an example of reform mathematics teaching.

What is most striking about this lesson, as well as the other instances of negotiate, is that the teacher engaged in learning during instruction. Furthermore, this learning was a critical factor in enabling her to manage the implementation of reform. For example, as she learned more about students' understandings, the teacher became better able to respond to students' questions and to decide on the appropriate direction for the lesson. In a similar manner, new understandings of mathematics and of the goals of the lesson aided the teacher's efforts to rely on reform-based teaching practices.

Lesson Plan for the Real-Staircases Lesson

When the real-staircases lesson occurred, students had not yet been introduced to the formal definition of slope as $\frac{\text{rise}}{\text{run}}$; however, prior to this lesson, students had investigated slope qualitatively for several days as part of the "starburst" lesson. In the starburst lesson students tried to create a starburst of lines on the computer by using different values for m in the equation $y = mx$ (see Figure 1). The goal of the activity was for students to explore how changing the value of m affected the inclination of the line. As a result, most students came to recognize that steep lines have big slopes and flat lines have small slopes and that lines with positive values for m rise to the right and lines with negative values for m fall to the right (Magidson, 1992).

For homework the night before the real-staircases lesson the students were asked to measure the rise and the run of a real staircase. The lesson plan explained that, the next day in class, groups of students would compare their staircase measurements and choose the steepest and flattest staircases within their group to reproduce on large grid paper (see Figure 2). These large staircases would then be displayed throughout the classroom. A class discussion would follow in which the groups would explain the criteria they had used to determine the steepness of a

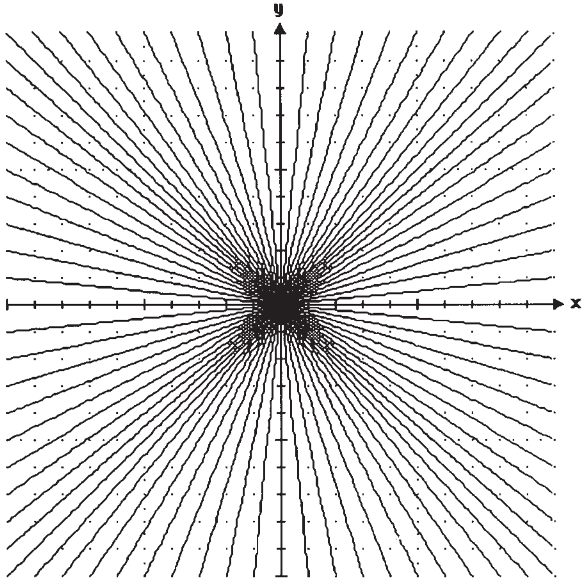

FIGURE 1 Can you make a starburst?

staircase. With these criteria in mind, the students would then select the steepest and flattest staircases from those displayed in the classroom. Finally, the students would try to figure out a way to determine the steepest staircase by working only with the values of the rise and the run of the different staircases.

The intended goal of this class discussion was twofold. First, students should understand that both rise and run are factors in determining the steepness of a staircase. Second, students should have the opportunity to propose different ways of combining rise and run to quantify the steepness of a staircase—and, through this, to come up with the formula that slope = $\frac{\text{rise}}{\text{run}}$. The lesson plan explained that, in addition to $\frac{\text{rise}}{\text{run}}$, students might propose that steepness is determined by rise + run, rise – run, or rise ×

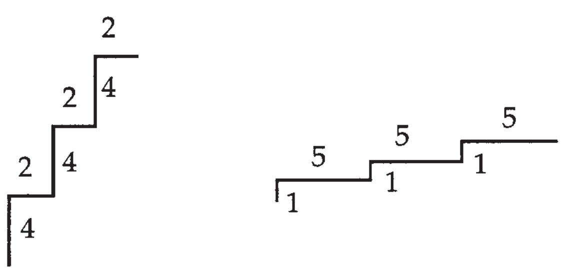

FIGURE 2 A steep staircase and a flat staircase.

run. In such cases, the lesson plan suggested that the teacher go through each of the proposed methods and ask the students whether the method works for all the staircases displayed or whether they can come up with a counterexample for the method. Thus, the idea was to use students' examples of real staircases to develop the formal relation that the steepness of a staircase depends on the ratio $\frac{\text{rise}}{\text{run}}$.

The Real-Staircases Lesson in Context

In Lynn's class the real-staircases lesson began much the way it was described in the lesson plan. As the lesson progressed, however, novel student ideas prompted her to develop new understandings of the domain and to adapt the lesson in line with these new ideas. Table 2 summarizes the negotiations among these areas of the teacher's content knowledge; it is intended to serve as a reference for readers during the next four sections of the article.

Cycle 1: A Counterexample to the Claim That Rise Determines Steepness

Lynn's students came to class with measurements for the rise and run of different staircases they had found in school, outside, and at home. Working in small groups, the students chose the steepest and flattest staircases from those that had been measured by members of their group. Each group reproduced these two staircases on large grid paper, and these were displayed throughout the classroom—the steepest staircases on one wall and the flattest staircases on another.

Lynn then asked each group to explain the criteria they had used to select a steepest staircase and a flattest staircase. Several groups claimed that the staircase with the largest rise was the steepest. A few students suggested that the length of the run determined the flatness of a staircase. Other students considered both the rise and the run as factors. They described steep staircases as those with a rise that was bigger than the run and flat staircases as those with a run that was bigger than the rise. In addition, one group explained that they simply looked at the drawings of their staircases and compared them visually to select the steepest and flattest staircases. During this part of the discussion Lynn restated each group's idea to clarify the different methods that students had used to determine steepness.

Next, Lynn asked the students to focus on the set of steep staircases hanging on the side wall and to decide which was the steepest. After some discussion, the class took a vote. Some students chose the staircase with a rise of 16 and a run of 17, whereas other students selected the staircase with a rise of 10 and a run of 7 (see Figure 3). In trying to help the class reach a consensus, Lynn asked students who supported each of the two positions to explain their reasoning. One student explained that if the rise is bigger than the run, then the staircase is steep. Thus, the staircase with rise 10 and run 7 was the steepest. In contrast, another student claimed that the length of the rise is all

TABLE 2
The Real Staircases Lesson: Four Cycles of Negotiations

Cycle	*Rethinking Students' Understanding*	*Rethinking the Domain*	*Rethinking the Lesson*
1	Students claim that the rise alone determines steepness.	As either the rise or run is varied, the slope will vary.	Cannot respond with slope = $\frac{\text{rise}}{\text{run}}$. Provide a counterexample for students' "rise only" claim.
2	Students suggest that if the rise > run, then the staircase is steep.	Comparing values for the rise and run is different than combining these values into a single quantity.	Focus on steepness as a single entity. Introduce the idea of building a ramp on a staircase.
3	Students suggest that if the rise < run, then the staircase is flat.	Comparing the rise and run differentiates between flat and steep staircases. Comparison does not depend on visualizing the staircases.	Focus on differentiating between flat and steep staircases. Ask students to decide whether a staircase is flat or steep using only the values of the rise and run.
4	Students have not come up with the formula that steepness = $\frac{\text{rise}}{\text{run}}$.	The slopes of lines are related to the steepness of staircases.	Modify familiar way to introduce slope = $\frac{\text{rise}}{\text{run}}$. Introduce the steepness of staircases as the slopes of line in the starburst.

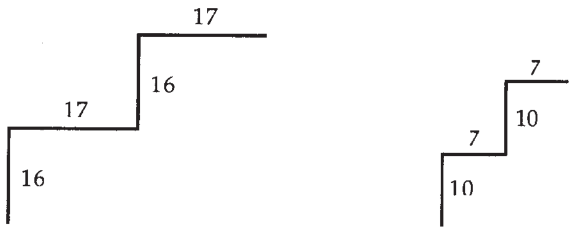

FIGURE 3 Choosing the steepest staircase.

that matters and that the staircase with the highest rise is the steepest. Therefore, this student believed that the staircase with rise 16 was the steepest.

At this point, Lynn introduced a new staircase into the discussion to help students see that both rise and run affect steepness. Because a rise of 16 was much larger than the rises of the other staircases on display, Lynn believed that this was a strong influence on the student's claim that rise alone determines steepness. She hypothesized that a rise of 16 would not appear to be as large to the students if the run were significantly larger than 16. Thus, she asked the class "What if the rise was still 16, but the run was much longer than 17?" The grid paper used by the students was not big enough for Lynn to sketch the staircase she had in mind; therefore, she used her hand to follow the rise and run of a flat staircase with a very long run. After presenting this counterexample she asked the students if they could now agree that both rise and run are factors in determining steepness. She then returned to the question of how to use the measurements of the rise and run to determine steepness.

As an aside, it is worth noting that it is not altogether surprising that some students claimed that the rise was the single determinant of steepness. In early work on proportional reasoning, Noelting (1980) identified an initial stage in students' development in which they examined only the first terms of two ratios that they were asked to compare. In a more recent study, Lobato and Thanheiser (1999) found that high school algebra students focused on a single attribute in examining steepness and, similar to the results reported here, granted greater status to height when describing the steepness of a ramp. Furthermore, as Lobato (1996) explained, this strategy may be particularly appealing to students when they are examining the steepness of staircases. Students may relate the term *steeper* with "harder to climb" and believe that it can be harder to climb a staircase with taller steps.

Negotiations within the teacher's content knowledge. Although it was relatively early in the lesson, Lynn was already negotiating among aspects of her content knowledge. In particular, she drew from her understanding of student learning, from her own understanding of the domain, and from her understanding of the lesson to develop new content knowledge and decide how to proceed. This negotiation began as she recognized a novel student idea—rise determines the steepness—

that was not in line with the goals of the lesson. As Lynn explained in the video club meeting after the lesson, her students had never suggested the rise-alone strategy in the past, and she was surprised that so many students offered this strategy: “I didn’t think quite so many kids would look at just one thing.” Lynn’s students were reasoning about steepness in a way that had not come up before.

With this in mind, Lynn tried to understand the mathematical basis for the students’ idea. She explained in the video club meeting that, during class, she realized that the rise alone strategy did in fact predict steepness for a subset of the staircases that the students were examining: “It worked [for some] of what they had to look at.” Nevertheless, Lynn understood that both the rise and the run determined steepness, and she wanted students to see this, to “understand that you needed both things.” Thus she considered her understanding of the students’ current thinking about the domain as well as her own knowledge of mathematics.

Lynn also turned to her view of the lesson to make sense of the situation. In particular, she recognized that her familiar ways for teaching about steepness did not provide her with an adequate response to the students’ claim. In her previous teaching experiences, the formula slope = $\frac{\text{rise}}{\text{run}}$ was given to the students as soon as they started working with slope. Therefore, in the past, if a student had suggested that the slope was determined by the rise, Lynn could have simply reminded the student that slope = $\frac{\text{rise}}{\text{run}}$, and not slope = rise. In the video club meeting she described such a response:

> If something like that came up, well then I would go off into this whole thing. I’d probably put those ordered pairs up on the graph. And try to show, using the overhead and the graph, how when you subtract the y’s that that gives you the vertical changes. And how when you subtract the x’s that gives you the horizontal change, and that’s why [rise over run] works.

Lynn continued by emphasizing that, for the purposes of the real-staircases lesson, she could not rely on that response because she “didn’t want to give away certain pieces of the plot too early.” In other words, she knew that discussing the formula $\frac{\text{rise}}{\text{run}}$ did not make sense at this point of the lesson. Instead, during instruction, Lynn developed a new pedagogical response appropriate for the situation at hand.

My claim is that Lynn learned something here. Through negotiations with her understanding of the students’ strategy, her knowledge of the mathematics involved, and her understanding of what the current lesson calls for, she learned. After watching the previously described excerpt during the video club meeting, Lynn stated that she had been nervous about responding to the strategies that students were expected to raise in class during this part of the lesson:

> I was all worried about trying to figure out … how to use the kids’ examples to show why things didn’t work … That made me nervous because I didn’t know what the kids were going to come up with. I couldn’t plan ahead.

However, she continued by explaining that, after class, she had a much better idea of the range of ideas that students might raise: "So then second period I had some idea of how it was going to work." Furthermore, in describing her sixth-period class she stated, "I expected ['rise only'] to come up ... I was kind of hoping it would because it would be good to talk about why it doesn't work." Lynn's statements provide evidence of her learning: She had developed a new pedagogical strategy and felt better equipped to discuss the "rise alone" strategy with her students.

Note that in this cycle Lynn's negotiations did not result in a change in the direction of the lesson; rather, she simply inserted the new counterexample into the lesson and then continued with the lesson as planned. Therefore, in this case her negotiation with the lesson is more of a negotiation with previous lessons she has taught on this topic—and the recognition that the current lesson calls for new ways to discuss steepness with students.

Cycle 2: Building a Ramp

As the lesson continued, Lynn reminded the class that they had decided that both rise and run were factors in determining steepness. She explained that what they needed to do next was to decide how the measurements of rise and run determine a staircase's steepness. She said, "So can we agree that steepness has to do with how far it goes up and how far it goes over? So then, we have to figure out, what's that relationship? How does it make it steep?"

In responding to Lynn's question, a number of students returned to the idea that the staircase in which the rise was much bigger than the run was the steepest. Lynn recognized that although the students were talking about the relation between the rise and the run, they had not modeled it with an operation. For example, they were not exploring whether rise + run or rise × run would help them to determine steepness. In contrast, they were simply focusing on which of the two values, the rise or the run, was bigger than the other.

With this understanding of the students' comments in mind, Lynn implemented a new pedagogical strategy to help students view steepness as a single quantity. She suggested that the students imagine building a ramp to adapt a staircase for a wheelchair. Lynn placed a large ruler against one of the staircases to illustrate such a ramp, and as she moved the ruler from staircase to staircase she asked the students to consider which ramp was the steepest.

The shift in perspective that Lynn was attempting to engender among her students has been described in the mathematics education literature as the *process of reification* (Kaput, 1987; Kieren, 1992; Sfard, 1992). The idea is that a mathematical concept can be viewed either as a process or as an object. For example, young children initially view the characters one, two, and three as elements of the counting song, that is, as part of a process. Later, these characters become entities in their own right that can be ordered and counted (Labinowicz, 1980). In a similar manner, at first students often view arithmetic expressions as directions to perform a calculation and only later recognize that these expressions are representations of specific numbers (Herscovics & Kieran, 1980).

A similar case has been made with functions (Moschkovich et al., 1993; Sfard, 1992). In some cases, students view a function as a process that links x-values and y-values. For example, the equation $y = x + 2$ can be seen as a function that connects every input value to an output value that is 2 more than the input. In other situations, students consider representations of functions as objects with global properties such as location and orientation. In this way, a graph of the line $y = x + 2$ can be thought of as the line that goes through the point (0, 2) and has slope of 1. These two different perspectives enable people to work with functions in very different ways, although switching between them has been shown to be problematic for students (Moschkovich et al., 1993).

In Lynn's class, students displayed what can be thought of as the *process view* of steepness. Specifically, they were comparing the individual attributes of steepness with steepness thought of as a process in which the rise is compared to the run. In contrast, Lynn's use of a ramp alongside a staircase illustrates the *object view* of steepness. Here, steepness itself is the attribute under examination, and students are asked to compare the steepness of various staircases.

Negotiations within the teacher's content knowledge. In this cycle of negotiation the teacher again interpreted an idea from the students and compared it to her understanding of the mathematics involved and to her understanding of the purpose of the lesson. Lynn initially recognized that the strategy her students were using was not a means of combining the rise and run into a single value to represent steepness. Instead, her students were simply characterizing steepness by comparing the rise and run of a staircase. In the video club meeting, Lynn explained that "they were using greater than or less than instead of an operation." In analyzing this strategy, the teacher drew from her knowledge of student learning in this domain and from her own knowledge of the subject matter.

In addition, Lynn reconciled these two perspectives on steepness with the goal of the lesson. She stated that "what I had trouble with was that kids would say, 'well if the rise is more than the run, then it's steeper' … [How could] I explain that [this strategy] wasn't an option [for determining steepness?]" As this quote illustrates, Lynn understood that her students were looking at steepness in a way that was different from what was expected in the lesson. Furthermore, Lynn hoped to move the students away from comparing the rise and the run and toward a focus on the idea that a single entity can represent the steepness of a staircase. Specifically, she attempted to illustrate this view of steepness using the idea of a ramp alongside a staircase. The idea of building a ramp was suggested in the lesson plan, but as an activity to be used earlier in the lesson for a different purpose.

In this cycle, Lynn negotiated with her understanding of what students were learning, with the mathematics that was under consideration, and with the outcome of the lesson. Once again I claim that Lynn has learned something through this negotiation process. She came to understand that students viewed steepness differently than intended, and she developed a new instructional strategy designed to redirect the students' thinking in a particular way. Furthermore, throughout the remainder of the unit Lynn continued to rely on the idea of ramps alongside staircases as a valuable technique for considering steepness as a single entity.

Cycle 3: Distinguishing Between Steep and Flat Staircases

After the discussion of steep staircases, Lynn returned to the lesson plan and asked the class to select the flattest staircase from those displayed. The students agreed on the flattest staircase and explained that this staircase had a rise that was much smaller than the run. Lynn again used the idea of ramps to try to encourage students to view steepness as a single entity, but the students persisted in comparing the rise and run to determine steepness.

At this point, Lynn recognized that although the students as a group were not moving toward quantifying steepness, they were able to distinguish between steep and flat staircases. Although the purpose of the real-staircases lesson was for students to find a way to compare staircases relative to each other, Lynn's students had instead discovered a way to organize staircases into two groups: steep staircases and flat staircases. To investigate this method further, Lynn developed a new instructional strategy.

Lynn: If I gave you some numbers and I asked you not to draw the picture of the staircase, but I just gave you the dimensions, the rise and run. Could you figure out how steep the staircase could be? Could you figure out whether it was steep or flat?

Students: Yeah.

Lynn: Let's try it. I'm thinking of a staircase that's 4 inches high, and it's 12 inches, the run is 12 inches. So the rise is 4 inches and the run is 12 inches. Don't draw a picture. Would that be on the flat side or the steep side?

Students: Flat.

Lynn asked the students to try to determine whether a staircase was steep or flat given the measurements of the rise and the run and without drawing the staircase. She gave the measurements aloud and asked students to imagine the staircase. Her first example was a rise that was smaller than the run. The students recognized the values as those of a flat staircase. Lynn then asked about a staircase that had a rise of 4 and a run of 4. A student explained that it is "right in the middle" and that it would be the same steepness as a staircase with a rise of 6 and run of 6. Lynn agreed, and then for her final example she gave the measurements of a steep staircase. Although this activity was not suggested in the lesson plan, it capitalized on the students' idea that they could distinguish between steep and flat staircases by comparing the rise and the run. In addition, it brought up the idea of a staircase being neither flat nor steep but "right in the middle."

Negotiations within the teacher's content knowledge. In this part of the lesson Lynn continued to negotiate among aspects of her content knowledge. First, she interpreted the students' strategy, explaining in the video club meetings that "The kids would say, 'If the rise is more that the run, then it's steeper. And if the run is more than the rise then it's flatter.'" In addition, she examined their strategy in terms of the underlying mathematical concepts. Lynn recognized that her students had developed an innovative and valuable way to work with steep and flat staircases: "They didn't get the idea that slope was rise over run, but they got the idea of how to make a staircase steeper and flatter by changing things." Lynn also negotiated with her understanding of the lesson, deciding that the students' method, although not a strategy suggested in the lesson plan, was worth pursuing. As a result, she shifted the direction of the lesson in response to the students' ideas.

Once again Lynn has learned something through this interaction. In particular, she learned a new way of thinking about steepness by comparing the rise and the run, and she understands that this comparison strategy, if done correctly, can help students explore an important relation among staircases. Furthermore, this is precisely the sort of knowledge that could be helpful to her in future attempts to teach the lesson. As I show in the next section, Lynn continued to draw on this knowledge as the lesson continued.

Cycle 4: Introducing $\frac{\text{rise}}{\text{run}}$

At this point in the lesson Lynn recognized that the students had developed some important ideas about the difference between steep and flat staircases. However, she also realized that they were not about to discover the ratio $\frac{\text{rise}}{\text{run}}$. Lynn intended to introduce the class to $\frac{\text{rise}}{\text{run}}$ that day, because that was the stated goal of the lesson and because students needed the formula to complete the night's homework. Therefore, with the class period close to ending, Lynn developed a new way to present $\frac{\text{rise}}{\text{run}}$.

Lynn began by explaining to the class that they wanted a single number to describe the steepness of a staircase so that, as a student had suggested, a staircase with a rise of 4 and a run of 4 would have the same steepness as a staircase with a rise of 6 and run of 6. Lynn reminded the students that in the starburst lesson they had called the steepness of lines the slope and that staircases also have slope. She explained that each staircase could be associated with a ramp and that the slope of the ramp is the steepness of the staircase.

Lynn: What we need is some way to put a number on steepness, because what Lena said really made sense that a staircase that was 4 inches high and 4 inches deep should be about the same as one that's 6 inches high and 6 [inches] deep, as far as the steepness goes. So, what we need is a way to quantify this. We already have a word to describe how steep a line was. Remember when we did starburst, we talked about lines with different, do you remember the word?

Students: Slopes.

Lynn: Right. We talked about lines with different slopes. Staircases can be defined by slopes also. If you think of staircases as being associated with a ramp that you would build on it for a wheelchair. Since I'm running low on time, what we want is a number that we can [use to] say, this line is steeper than this line, or this staircase is steeper than that staircase.

Lynn then reminded the students of the different regions of the plane that they had discussed after creating the starburst (see Figure 4). Specifically, the class had investigated the values of the slopes of lines that pass through the origin. Those lines with slope values between 0 and 1 lie between the *x*-axis and the line $y = x$. Similarly, lines with slopes between 1 and infinity lie between the line $y = x$ and the *y*-axis. Lynn now pointed out that those same regions distinguish between flat and steep lines and, similarly, between the associated flat and steep staircases.

Finally, Lynn explained that, to find the slope number for a staircase, the students needed to divide the rise by the run. She then calculated $\frac{\text{rise}}{\text{run}}$ for some of the students' staircases and determined the region in which each staircase belonged.

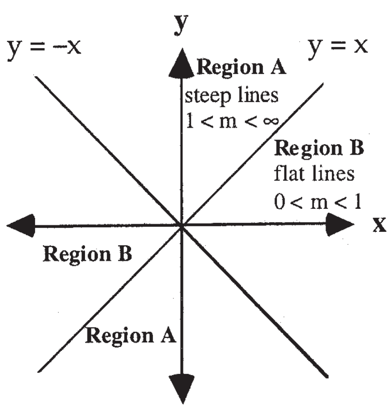

FIGURE 4 Regions that distinguish between steep and flat lines.

Lynn: One way to compare slope is to compare the rise to the run. This staircase has a rise of 10 and a run of 7. If we divide 10 by 7, do you get a number less than 1 or more than 1?

Students: More.

Lynn: You get a number more than 1, don't you, 10 divided by 7. So that would be in Region A. So that's a steeper staircases than say, this one. [Lynn points to the staircase with a rise of 3 and a run of 6.]

Student: It would be half of it.

Lynn: If you look at this one, the rise is 3 and the run is 6. What's 3 divided by 6? Is that smaller than 1 or more than 1?

Students: Smaller.

Lynn: So the line that we could draw with this staircase would be in Region B if we did it. It would be smaller than 1. So it's a flatter one.

Negotiations within the teacher's content knowledge. In this final section of the real-staircases lesson Lynn engaged in a fourth cycle of negotiations. She first considered the goals of the lesson along with her understanding of what students had learned in the lesson thus far. She recognized that although $\frac{\text{rise}}{\text{run}}$ was intended to be the central focus of the lesson, the students had not discovered this ratio on their own. Furthermore, her familiar way of introducing this concept—simply telling students that slope = $\frac{\text{rise}}{\text{run}}$—would not draw on the ideas with which students had worked during the lesson. Thus, she chose not to implement this familiar pedagogical routine.

Lynn also realized that the method suggested in the lesson plan for coming up with $\frac{\text{rise}}{\text{run}}$ did not materialize during implementation. According to the lesson plan, students would have explored various ways to combine the rise and run, such as rise + run and rise × run, before concluding that $\frac{\text{rise}}{\text{run}}$ determines the steepness of a staircase. Instead, with Lynn's assistance, the students took a very different path in exploring steepness. As a result, Lynn recognized the need to develop a new way to introduce $\frac{\text{rise}}{\text{run}}$.

As in the previous cycles of negotiations, Lynn's new pedagogical strategy reflected her understanding of the mathematics involved, the students' ideas that had come up during the lesson, and what the students were supposed to understand by the end of the class period. In particular, Lynn explicitly connected the steepness of staircases with the slopes of lines and used the idea of a ramp to bridge the day's lesson with the starburst activity. In addition, she connected the lesson's ultimate focus on steep and flat staircases to the regions discussed in the starburst lesson.

A little context may help the reader understand the magnitude of Lynn's accomplishment here. Simon and Blume (1994) examined the ability of preservice teachers to understand a ratio as the measure of a given attribute such as steepness, what they call an understanding of *ratio-as-measure*. They found that this concept was quite challenging for the teachers in their study. Furthermore, by examining the difficulties that the teachers encountered, Simon and Blume concluded that understanding ratio-as-measure involves an understanding of multiplicative relations as well as an understanding of mathematical modeling.

This analysis of the real-staircases lesson illustrates that Lynn had an understanding of ratio-as-measure and that she meaningfully applied this understanding to construct the day's lesson. First, in terms of the multiplicative relation being discussed, Lynn connected the ratio $\frac{\text{rise}}{\text{run}}$ to the idea of steep versus flat staircases. In particular, she illustrated that for steep staircases this ratio has a value greater than 1 and that for flat staircases the ratio has a value less than 1. Furthermore, she developed a mathematical model of the situation for the class by (a) connecting the slope of a line to the steepness of a ramp alongside a staircase and (b) connecting steep and flat staircases to particular regions of the Cartesian plane.

Of course, unlike the preservice teachers in Simon and Blume's (1994) study, Lynn was aware of the formula slope = $\frac{\text{rise}}{\text{run}}$ at the start of the lesson—she was not asked to create this ratio from scratch. In addition, she is a secondary mathematics teacher; those in Simon and Blume's study were elementary school teachers. Thus, one might expect Lynn to have greater subject matter expertise. However, in interviews prior to teaching the unit, Lynn did not demonstrate a ratio-as-measure understanding of slope. For example, when asked to discuss the meaning of slope, Lynn did describe slope as a measure of steepness and cited the formula slope = $\frac{\text{rise}}{\text{run}}$. Yet she offered these as two different conceptions of slope and, even when prompted, proposed no connections between them. In addition, prior to the unit, Lynn tended to discuss the slope as the number m in the equation $y = mx + b$, explaining that "the bigger the number, the steeper the line." She never discussed m as being determined by a ratio of two numbers, and neither did she explore how changes in either the rise or the run would affect a line's steepness. In other words, aside from naming the slope as $\frac{\text{rise}}{\text{run}}$, Lynn did not explicitly consider the multiplicative properties of this ratio. Although it may be the case that Lynn did in fact have a ratio-as-measure understanding of slope, she did not discuss these ideas in the interviews.

This stands in stark contrast to the ideas on which Lynn drew in this final cycle of the real-staircases lesson. Here she connected steepness and slope and used the ratio of rise to run to help students see this connection. It appears that during the negotiation process, Lynn enriched her understanding of ratio-as-measure. Furthermore, as Lynn continued to work with her students to examine slopes and staircases in the rest of the unit, there is evidence that she now drew on a fairly robust understanding of ratio-as-measure. In particular, she consistently emphasized with her students that changing either the rise or the run influences the steepness of lines in specific ways. Thus, there is some evidence that Lynn learned in a substantial way.

CONCLUSIONS

The real-staircases lesson illustrates the application of content knowledge during the implementation of reform. In particular, the example demonstrates how negotiations among aspects of the teacher's content knowledge enabled her to manage this complex form of instruction. Furthermore, in doing so the teacher engaged in learning: She modified her existing content knowledge and developed new understandings of the domain and new pedagogical strategies appropriate for the current context.

The previous example also provides insights into the process by which these cycles of negotiations occur. I now address three issues related to this process. For each issue, I draw connections between the specific negotiations described in the example and a more general conception of the manner in which teachers apply their content knowledge during the implementation of reform.

Novel Student Behavior as a Trigger for Change in Teachers' Content Knowledge

Earlier in this article I introduced the notion that teachers have knowledge structures called content knowledge complexes. I argued that, because of the strong connections between pedagogy and content that exist within these structures, teachers almost automatically implement their established pedagogical routines. Yet, as one can see in the real-staircases lesson, during the implementation of reform teachers may apply their content knowledge differently. How does this happen?

In brief, I find that novel student comments or methods are a critical factor in provoking teachers to move away from the constraints of their content knowledge complexes. For example, in the first cycle of negotiations in the staircases lesson Lynn attempted to respond to the students' idea that the rise determines steepness. Because she recognized that her familiar ways of thinking about and teaching this topic—that slope = $\frac{\text{rise}}{\text{run}}$—did not provide her with an appropriate response in this context, she looked beyond her content knowledge complex related to this topic to decide how to respond.

Other researchers have also suggested that listening to and learning to interpret students' ideas can be a key catalyst for teacher change (Franke et al., 1997; Heaton, 2000; Wood et al., 1991). It is worth noting that in the data for this study, in all instances of negotiate, the teachers moved away from an established content knowledge complex in response to a novel idea raised by a student during instruction. It is as though an unexpected or innovative comment from a student is a signpost for the teacher to check on the progress of the lesson. Moreover, rethinking their knowledge of student understanding can prompt teachers to rethink other areas of their content knowledge as well. Although other types of issues may serve as prompts for teachers' rethinking and learning during the planning phase of instruction (Remillard, 2000), novel student ideas have emerged as a key trigger for teacher learning during instruction.

An Elaboration of Pedagogical Reasoning

This research also offers a mechanism through which one can understand how teachers learn during the act of instruction. Previous research in this area introduced the idea of pedagogical reasoning as a way that novice teachers adapt their content knowledge for teaching (Shulman, 1987; S. M. Wilson, Shulman, & Richert, 1987). The idea is that pedagogical reasoning acts on subject matter knowledge to produce pedagogical content knowledge. Thus, the teacher starts with an understanding of some particular content and, by considering how to present the content to students, transforms this knowledge into pedagogical content knowledge.

This article illustrates another process through which teacher learning occurs. I have found that learning, rather than being a linear development that builds primarily on subject matter knowledge, can also occur as a cycle of negotiations that builds on both subject matter knowledge and on pedagogical content knowledge. Furthermore, this cycle of negotiations results in increased subject matter knowledge and pedagogical content knowledge on the part of the teacher.

Development of New Content Knowledge Complexes

I now raise one final issue for consideration. On the basis of the view of teacher learning presented here, I suggest that it is through the case of negotiate that teachers develop new content knowledge complexes. First, recall that as teachers negotiate among their knowledge of student thinking, mathematics, and the lesson, they develop new understandings in each of these areas. Furthermore, these new ideas about subject matter and about student learning become connected to the pedagogy of the lesson. This is precisely the sort of integration between subject matter knowledge and pedagogical content knowledge that characterizes a content knowledge complex. Thus, it seems plausible that, through the negotiation process, a new content knowledge complex is formed that functions in the same ways that earlier ones did.

I also hypothesize that, through the case of negotiate, new connections develop between existing and new content knowledge complexes and, as result, an existing content knowledge complex is refined. An important consequence of these changes is that the teacher can now consider his or her familiar teaching strategies without simply implementing the pedagogy suggested. For example, in the real-staircases lesson, Lynn's new pedagogical strategies for teaching about steepness formed ties to her familiar way of teaching slope. Therefore, at the end of the lesson, when she decides to introduce the formula slope = $\frac{\text{rise}}{\text{run}}$, Lynn does not automatically implement her familiar introduction to the formula; rather, she accesses her modified content knowledge complex and considers her familiar teaching strategies along with her new ideas about teaching slope.

When Teaching Becomes Learning

In this article I describe how, in the midst of a lesson, teachers negotiate among aspects of their content knowledge and adapt their instruction as a result. Because these on-the-spot negotiations involve changes in the teachers' understandings, I claim that they constitute learning for the teacher. Furthermore, I believe that the perspective of teaching as learning can be a powerful metaphor for teachers and researchers. Thus, it is not sufficient that teachers learn new ideas about the domain and new teaching practices to implement reform. Instead, the process of reform-based teaching is itself a learning process and should be recognized as such.

Therefore, in addition to examining the learning that teachers must do prior to instruction, researchers should investigate ways to promote the learning that can occur during instruction. In particular, if teachers can be helped to recognize opportunities to negotiate among aspects of their content knowledge during instruction, new ways to help teachers sustain the complex forms of instruction called for by reform may be found. Such research would add to current understandings of teachers' content knowledge and elucidate its role in effective teaching in times of change.

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