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THE NATURE OF MATHEMATICS: ITS ROLE AND ITS INFLUENCE

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Perceptions of the nature and role of mathematics held by our society have a major influence on the development of school mathematics curriculum, instruction, and research. The understanding of different conceptions of mathematics is as important to the development and successful implementation of programs in school mathematics as it is to the conduct and interpretation of research studies. The literature of the reform movement in mathematics and science education (American Association for the Advancement of Science, 1989; Mathematical Sciences Education Board, 1989, 1990; National Council of Teachers of Mathematics, 1989) portrays mathematics as a dynamic, growing field of study. Other conceptions of the subject define mathematics as a static discipline, with a known set of concepts, principles, and skills (Fisher, 1990).

The rapid growth of mathematics and its applications over the past 50 years has led to a number of scholarly essays that examine its nature and its importance (Consortium for Mathematics and Its Applications, 1988; Committee on Support of Research in the Mathematical Sciences, 1969; Courant & Robbins, 1941; Davis & Hersh, 1980, 1986; Hardy, 1940; Hilton, 1984; Saaty & Weyl, 1969; Steen, 1978; Stewart, 1987; Wilder, 1968). This literature has woven a rich mosaic of conceptions of the nature of mathematics, ranging from axiomatic structures to generalized heuristics for solving problems. These diverse views of the nature of mathematics also have a pronounced impact on the ways in which our society conceives of mathematics and reacts to its ever-widening influence on our daily lives. Regarding this, Steen (1988) writes:

> Many educated persons, especially scientists and engineers, harbor an image of mathematics as akin to a tree of knowledge: formulas, theorems, and results hang like ripe fruits to be plucked by passing scientists to nourish their theories. Mathematicians, in contrast, see their field as a rapidly growing rain forest, nourished and shaped by forces outside mathematics while contributing to human civilization a rich and ever-changing variety of intellectual flora and fauna. These differences in perception are due primarily to the steep and harsh terrain of abstract language that separates the mathematical rain forest from the domain of ordinary human activity. (p. 611)

Research shows that these differing conceptions have an influence on the ways in which both teachers and mathematicians approach the teaching and development of mathematics (Brown, 1985; Bush, 1982; Cooney, 1985; Good, Grouws, & Ebmeier, 1983; Kesler, 1985; McGalliard, 1983; Owens, 1987; Thompson, 1984). Some see mathematics as a static discipline developed abstractly. Others see mathematics as a dynamic discipline, constantly changing as a result of new discoveries from experimentation and application (Crosswhite et al., 1986). These contrasting views of the nature and source of mathematical knowledge have provided a continuum for conceptions of mathematics since the age of the Greeks. The lack of a common philosophy of mathematics has serious ramifications for both the practice and teaching of mathematics. This lack of consensus, some argue, is the reason that differing philosophies are not even discussed. Others conjecture that these views are transmitted to students and help shape their ideas about the nature of mathematics (Brown, Cooney, & Jones, 1990; Cooney, 1987). What follows is an overview of these conceptions of mathematics and their current and potential impact on the nature and course of mathematics education.

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CONCEPTIONS OF MATHEMATICS

Historical

Discussions of the nature of mathematics date back to the fourth century B C. Among the first major contributors to the dialogue were Plato and his student, Aristotle. Plato took the position that the objects of mathematics had an existence of their own, beyond the mind, in the external world. In doing so, Plato drew clear distinctions between the ideas of the mind and their representations perceived in the world by the senses. This caused Plato to draw distinctions between arithmetic—the theory of numbers—and logistics—the techniques of computation required by businessmen. In the *Republic* (1952a), Plato argued that the study of arithmetic has a positive effect on individuals, compelling them to reason about abstract numbers. Plato consistently held to this view, showing indignation at technicians' use of physical arguments to "prove" results in applied settings. For Plato, mathematics came to "be identical with philosophy for modern thinkers, though they say that it should be studied for the sake of other things" (Aristotle, 1952, p. 510). This elevated position for mathematics as an abstract mental activity on externally existing objects that have only representations in the sensual world is also seen in Plato's discussion of the five regular solids in *Timaeus* (1952b) and his support and encouragement of the mathematical development of Athens (Boyer, 1968).

Aristotle, the student, viewed mathematics as one of three genera into which knowledge could be divided: the physical, the mathematical, and the theological:

> [Mathematics is the one] which shows up quality with respect to forms and local motions, seeking figure, number, and magnitude, and also place, time, and similar things.... Such an essence falls, as it were, between the other two, not only because it can be conceived both through the senses and without the senses. (Ptolemy, 1952, p. 5)

This affirmation of the role of the senses as a source for abstracting ideas concerning mathematics was different from the view held by his teacher, Plato. Aristotle's view of mathematics was not based on a theory of an external, independent, unobservable body of knowledge. Rather it was based on experienced reality, where knowledge is obtained from experimentation, observation, and abstraction. This view supports the conception that one constructs the relations inherent in a given mathematical situation. In Aristotle's view, the construction of a mathematical idea comes through idealizations performed by the mathematician as a result of experience with objects. Thus, statements in applied mathematics are approximations of theorems in pure mathematics (Körner, 1960). Aristotle attempted to understand mathematical relationships through the collection and classification of empirical results derived from experiments and observations and then by deduction of a system to explain the inherent relationships in the data. Thus, the works and ideas of Plato and Aristotle molded two of the major contrasting themes concerning the nature of mathematics.

By the Middle Ages, Aristotle's work became known for its contributions to logic and its use in substantiating scientific claims. Although this was not contrary to the way in which Aristotle had employed his methods of logical reasoning, those who employed his principles often used them to argue against the derivation of evidence from empirical investigations. Aristotle drew clear lines between the ideal *forms* envisioned by Plato and their empirical realizations in worldly objects.

The distinctions between these two schools of mathematical thought were further commented upon by Francis Bacon in the early 1500s when he separated mathematics into pure and mixed mathematics:

> To the pure mathematics are those sciences belonging which handle quantity determinate, merely severed from any axioms of natural philosophy.... For many parts of nature can neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity, nor accommodated unto use with sufficient dexterity, without the aid and intervening of the mathematics. (1952, p. 46)

Similar discussions concerning the nature of mathematics were also echoed by Jean D'Alembert and other members of the French salon circle (Brown, 1988).

Descartes worked to move mathematics back to the path of deduction from accepted axioms. Though experimenting himself in biological matters, Descartes rejected input from experimentation and the senses in matters mathematical because it might possibly delude the perceiver. Descartes's consideration of mathematics worked to separate it from the senses:

> For since the name "Mathematics" means exactly the same as "scientific study,"... we see that almost anyone who has had the slightest schooling, can easily distinguish what relates to Mathematics in any question from that which belongs to the other sciences.... I saw consequently that there must be some general science to explain that element as a whole which gives rise to problems about order and measurement restricted as these are to no special subject matter. This, I perceived, was called "Universal Mathematics," not a far fetched designation, but one of long standing which has passed into current use, because in this science is contained everything on account of which the others are called parts of Mathematics. (1952, p. 7)

This struggle between the rationalists and the experimentalists affected all branches of science throughout the 17th and 18th centuries.

The German philosopher Immanuel Kant brought the discussion of the nature of mathematics, most notably the nature of geometry, back in to central focus with his *Critique of Pure Reason* (1952). Whereas he affirmed that all axioms and theorems of mathematics were truths, he held the view that the nature of perceptual space was Euclidean and that the contents of Euclidean geometry were a priori understandings of the human mind. This was in direct opposition to the emerging understandings of non-Euclidean geometry.

The establishment of the consistency of non-Euclidean geometry in the mid-1800s finally freed mathematics from the restrictive yoke of a single set of axioms thought to be the only model for the external world. The existence of consistent non-Euclidean geometries showed the power of man's mind to construct new mathematical structures, free from the bounds of an externally existing, controlling world (Eves, 1981; Kline, 1972, 1985; Körner, 1960). This discovery, exciting as it was, brought with it a new notion of "truth," one buried in the acceptance of an axiom or a set of axioms defining a model for an area of investigation. Mathematicians immediately began to apply this new freedom and axiomatic method to the study of mathematics.

Late 19th and Early 20th Century Views

New investigations in mathematics, freed from reliance on experimentation and perception, soon encountered new problems with the appearance of paradoxes in the real number system and the theory of sets. At this point, three new views of mathematics arose to deal with the perceived problems. The first was the school of logicism, founded by the German mathematician Gottlob Frege in 1884. This school, an outgrowth of the Platonic school, set out to show that ideas of mathematics could be viewed as a subset of the ideas of logic. The proponents of logicism set out to show that mathematical propositions could be expressed as completely general propositions whose truth followed from their form rather than from their interpretation in a specific contextual setting. A. N. Whitehead and Bertrand Russell (1910–13) set out to show this in their landmark work, *Principia Mathematica*. This attempt was equivalent to trying to establish classical mathematics from the terms of the axioms of the set theory developed by Zermelo and Frankel. This approach, as that of Frege, was built on the acceptance of an externally existing mathematics, and hence was a direct outgrowth of the Platonic school. Whitehead and Russell's approach failed through its inability to establish the axioms of infinity and choice in a state of complete generality devoid of context. This Platonic approach also failed because of the paradoxes in the system.

The followers of the Dutch mathematician L. E. J. Brouwer, on the other hand, did not accept the existence of any idea of classical mathematics unless it could be constructed via a combination of clear inductive steps from first principles. The members of Brouwer's school of thought, called the intuitionists, were greatly concerned with the appearance of paradoxes in set theory and their possible ramifications for all of classical mathematics. Unlike the logicists, who accepted the contents of classical mathematics, the intuitionists accepted only the mathematics that could be developed from the natural numbers forward through the mental activities of constructive proofs. This approach did not allow the use of the law of the excluded middle. This logical form asserts that the statement $p \vee \sim p$ is true and makes proof by contradiction possible.

In many ways, the ideas put forth by Brouwer were based on a foundation not unlike that professed by Kant. Brouwer did not argue for the "inspection of external objects, but [for] 'close introspection"' (Körner, 1960, p. 120). This conception portrayed mathematics as the objects resulting from "valid" demonstrations. Mathematical ideas existed only insofar as they were constructible by the human mind. The insistence on construction placed the mathematics of the intuitionists within the Aristotelian tradition. This view took logic to be a subset of mathematics. The intuitionists' labors resulted in a set of theorems and conceptions different from those of classical mathematics. Under their criteria for existence and validity, it is possible to show that every real-valued function defined for all real numbers is continuous. Needless to say, this and other differences from classical mathematics have not attracted a large number of converts to intuitionism.

The third conception of mathematics to emerge near the beginning of the 20th century was that of formalism. This school was molded by the German mathematician David Hilbert. Hilbert's views, like those of Brouwer, were more in line with the Aristotelian tradition than with Platonism. Hilbert did not accept the Kantian notion that the structure of arithmetic and geometry existed as descriptions of a priori knowledge to the same degree that Brouwer did. However, he did see mathematics as arising from intuition based on objects that could at least be considered as having concrete representations in the mind.

Formalism was grounded in the attempts to characterize mathematical ideas in terms of formal axiomatic systems. This attempt to free mathematics from contradictions was built around the construction of a set of axioms for a branch of mathematics that allowed for the topic to be discussed in a first-order language. Considerable progress was made in several areas under the aegis of formalism before its demise as a result of Kurt Gödel's 1931 landmark paper. Gödel (1931) established that it is impossible in axiomatic systems of the type Hilbert proposed to prove formally that the system is free of contradictions. Gödel also demonstrated that it is impossible to establish the consistency of a system employing the usual logic and number theory if one uses only the major concepts and methods from traditional number theory. These findings ended the attempt to so formalize all of mathematics, though the formalist school has continued to have a strong impact on the development of mathematics (Benacerraf & Putnam, 1964; van Heijenoort, 1967; Snapper, 1979a, 1979b).

The three major schools of thought created in the early 1900s to deal with the paradoxes discovered in the late 19th century advanced the discussion of the nature of mathematics, yet none of them provided a widely adopted foundation for the nature of mathematics. All three of them tended to view the contents of mathematics as products. In logicism, the contents were the elements of the body of classical mathematics, its definitions, its postulates, and its theorems. In intuitionism, the contents were the theorems that had been constructed from first principles via "valid" patterns of reasoning. In formalism, mathematics was made up of the formal axiomatic structures developed to rid classical mathematics of its shortcomings. The influence of the Platonic and Aristotelian notions still ran as a strong undercurrent through these theories. The origin of the "product"—either as a pre-existing external object or as an object created through experience from sense perceptions or experimentation—remained an issue.

Modern Views

The use of a product orientation to characterize the nature of mathematics is not a settled issue among mathematicians. They tend to carry strong Platonic views about the existence of mathematical concepts outside the human mind. When pushed to make clear their conceptions of mathematics, most retreat to a formalist, or Aristotelian, position of mathematics as a game played with symbol systems according to a fixed set of socially accepted rules (Davis & Hersh, 1980). In reality, however, most professional mathematicians think little about the nature of their subject as they work within it.

The formalist tradition retains a strong influence on the development of mathematics (Benacerraf & Putnam, 1964; Tymoczko, 1986). Hersh (1986) argues that the search for the foundations of mathematics is misguided. He suggests that the focus be shifted to the study of the contemporary practice of mathematics, with the notion that current practice is inherently fallible and, at the same time, a very public activity (Tymoczko, 1986). To do this, Hersh begins by describing the plight of the working mathematician. During the creation of new mathematics, the mathematician works as if the discipline describes an externally existing objective reality. But when discussing the nature of mathematics, the mathematician often rejects this notion and describes it as a meaningless game played with symbols. This lack of a commonly accepted view of the nature of mathematics among mathematicians has serious ramifications for the practice of mathematics education, as well as for mathematics itself.

The conception of mathematics held by the teacher has a strong impact on the way in which mathematics is approached in the classroom (Cooney, 1985). A teacher who has a formalist philosophy will present content in a structural format, calling on set theoretic language and conceptions (Hersh, 1986). Such a formalistic approach may be a good retreat for the individual who does not understand the material well enough to provide an insightful constructive view. Yet, if such formalism is *not* the notion carried by mathematicians, why should it dominate the presentation of mathematics in the classroom? To confront this issue, a discussion of the nature of mathematics must come to the foreground in mathematics education.

Tymoczko and Hersh argue that what is needed is a new philosophy of mathematics, one that will serve as a basis for the working mathematician and the working mathematics educator. According to Hersh, the working mathematician is not controlled by constant attention to validating every step with an accepted formal argument. Rather, the mathematician proceeds, guided by intuition, in exploring concepts and their interactions. Such a path places the focus on understanding as a guide, not long, formal derivations of carefully quantified results in a formal language.

This shift calls for a major change. Mathematics must be accepted as a human activity, an activity not strictly governed by any one school of thought (logicist, formalist, or constructivist). Such an approach would answer the question of what mathematics is by saying that:

> Mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be represented or suggested by physical objects). What are the main properties of mathematical activity or mathematical knowledge, as known to all of us from daily experience?
>
> 1. Mathematical objects are invented or created by humans.
> 2. They are created, not arbitrarily, but arise from activity with already existing mathematical objects, and from the needs of science and daily life.
> 3. Once created, mathematical objects have properties which are well-determined, which we may have great difficulty in discovering, but which are possessed independently of our knowledge of them. (Hersh, 1986, p. 22)

The development and acceptance of a philosophy of mathematics carries with it challenges for mathematics and mathematics education. A philosophy should call for experiences that help mathematician, teacher, and student to experience the invention of mathematics. It should call for experiences that allow for the mathematization, or modeling, of ideas and events. Developing a new philosophy of mathematics requires discussion and communication of alternative views of mathematics to determine a valid and workable characterization of the discipline.

TEACHERS' CONCEPTIONS OF MATHEMATICS

The conception of mathematics held by the teacher may have a great deal to do with the way in which mathematics is characterized in classroom teaching. The subtle messages communicated to children about mathematics and its nature may, in turn, affect the way they grow to view mathematics and its role in their world.

Cooney (1987) has argued that substantive changes in the teaching of mathematics such as those suggested by the NCTM *Standards* (1989) will be slow in coming and difficult to achieve because of the basic beliefs teachers hold about the nature of mathematics. He notes that the most prevalent verb used by preservice teachers to describe their teaching is *present*. This conception of teaching embodies the notion of authority in that there is a presenter with a fixed message to send. Such a position assumes the external existence of a body of knowledge to be transmitted to the learners and is thus more Platonic than Aristotelian. The extension of this conception of how mathematics relates to education and its practice is an important one. The teacher's view of how teaching should take place in the classroom is strongly based on a teacher's understanding of the nature of mathematics, not on what he or she believes is the best way to teach (Hersh, 1986). To change the situation, one must construct alternative ways of conceptualizing the nature of mathematics and the implications of such conceptions for mathematics education.

Cooney (1987) used the work of Goffree (1985) and Perry (1970) in his analyses of the nature of mathematics portrayed in classrooms and concluded that school mathematics is bound up in a formal and external view of mathematics. Goffree presented a model for the way textbooks are developed and how teachers might employ them in the classroom to portray the nature of mathematics. The four textbook models were (a) the mechanistic, (b) the structuralist, (c) the empiricist, and (d) the realistic or applied. Each of these methods of textbook development portrays a view of the nature of mathematics. Goffree then crossed these textual characteristics with three ways in which teachers employ textbooks in the classroom:

> Instrumental use—The teacher uses the textbook as an instrument, following its sequence and using its suggestions for dealing with the content.
>
> Subjective use—The teacher uses the textbook as a guide, but provides a constructive overview of the materials, followed by a further

discussion of the concepts/principles/procedures based on the teachers' experience.

Fundamental use—At this level, the curriculum is developed from a constructive viewpoint. This approach is concerned with both the content and pedagogy involved in mathematics. (p. 26)

In many classrooms, the prevailing model is mechanistic-instrumental. Modern reform documents (NCTM, 1989) advocate a situation that is closer to realistic-fundamental. The enormous distance between these two models indicates the large role that the teacher's conception of the nature of mathematics can play in the teaching and learning process as it applies to school mathematics.

In related work, Cooney (1985) and his students (Brown, 1985; Bush, 1982; Kesler, 1985; McGalliard, 1983; Owens, 1987; Thompson, 1984) have also examined the nature of teachers' conceptions of mathematics using the levels of intellectual development created by William Perry (1970). Perry's model provides a means to describe the way in which humans view the world about them. Perry's hierarchical scheme sees individuals passing through stages from dualism to multiplistic perspectives to relativistic perspectives. In the dualistic stage, the individual assumes that one functions in a bipolar world with such choices as good or bad, right or wrong. At this stage, problems are resolved by an authority's ruling. The individual may grow to a stage where multiple perspectives are entertained; however, the perspectives are still viewed as discrete entities lacking structural relationships. Finally, a person may move to the stage of relativism, where a number of possible alternatives are considered relative to one another. At this stage, each of the alternatives is examined within its own frame of reference.

Kesler (1985) and McGalliard (1983) conducted studies of secondary school algebra and geometry teachers' conceptions of mathematics by analyzing their classroom teaching. Kesler found that algebra teachers differed greatly in their orientations. Some performed at the dualistic or multiplistic level of the hierarchy, whereas others showed signs of multiplistic-relativistic behaviors. McGalliard's study of geometry teachers showed that their view of mathematics was marked by dualism. These teachers viewed their task as one of presenting mathematics to their students. The teachers' main concern was in seeing that their students learned to perform easily the tasks required by their homework and tests. Thus, the learning of mathematics was reduced to knowing how rather than knowing why. The fact that fewer teachers in geometry exceeded the dualistic level might be a reflection of their lack of geometric experience. Cooney (1987) reflects on the predominance and implications of the presenting, or broadcast, mode for teaching. Presenting, by its very nature, involves authority. Such an orientation is not compatible with a style of classroom management and resource use that would promote student consideration of a number of perspectives on mathematics, its nature, and its use. These ideas, plus collaborating findings by Owens (1987) with preservice secondary teachers, suggest the great distance that must be covered to bring the classroom consideration of mathematics close to the fundamental-realistic combination envisioned by Goffree.

Owens's work, and that of Bush (1982), further indicated that many of the preservice teachers' dualistic or multiplistic views were strengthened by their experiences in upper-division mathematics content courses at the university level. There, they were exposed to teaching that strongly reflected the formalist view of mathematics as an externally developed axiom system. This influence only reinforces the conception that mathematics exists externally. Through direct intervention, Myerson (1977) was able to move some students to view mathematics on a somewhat higher level. But many still thought that there were specific, set methods to address each classroom question, reflecting the strong dualistic-multiplistic orientation of preservice teachers.

The reaction of students is a strong factor influencing a teacher's portrayal of the nature of mathematics in class. Brown (1985) and Cooney (1985) studied the reactions of a first-year teacher in the classroom. The teacher entered the classroom with an orientation that reflected both multiplistic and relativistic characteristics. He attempted to initiate a classroom style involving a good deal of problem solving and student activities aimed at providing a strong foundation for student learning. The students found these approaches threatening and their reactions led to his eventual return to a presenting mode. Cooney (1987) concludes: "I suspect that students gravitate toward a mechanistic curriculum and appreciate teachers whose interpretations of the text are quite predictable. If you believe the contrary, listen carefully to the negotiations that take place between students and teacher when test time arrives" (p. 27).

THE RELEVANCE OF CONCEPTIONS OF MATHEMATICS TO MATHEMATICS EDUCATION RESEARCH

The focus on mathematics education and the growth of research in mathematics education in the late 1970s and the 1980s reflects a renewed interest in the philosophy of mathematics and its relation to learning and teaching. At least five conceptions of mathematics can be identified in mathematics education literature (J. Sowder, 1989). These conceptions include two groups of studies from the external (Platonic) view of mathematics. The remaining three groups of studies take a more internal (Aristotelian) view.

External Conceptions

The work of two groups of researchers treats mathematics as an externally existing, established body of concepts, facts, principles, and skills available in syllabi and curricular materials. The work of the first group of researchers adopting the external view focuses on assisting teachers and schools to be more successful in conveying this knowledge to children. Their work takes a relatively fixed, static view of mathematics.

Studies investigating the role of teachers in mathematics classrooms commonly focus on the actions and instructional methods of the teachers rather than on the mathematics being taught or the methods by which that mathematics is being learned. Early studies of teacher actions by B. O. Smith and his coworkers (Smith, Meux, Coombs, Nuthall, & Precians, 1967) led to a number of studies of the relative efficacy of the use of logical discourse in the teaching of concepts and generalizations (Cooney, 1980; Cooney & Bradbard, 1976; L. Sowder, 1980). Later research on effective teaching selected mathematics classrooms as the site for data gathering (Brophy, 1986; Brophy & Evertson, 1981; Brophy & Good, 1986; Fisher & Berliner, 1985; Good, Grouws, & Ebmeier, 1983; Medley & Mitzel, 1963; Rosenshine & Furst, 1973; Slavin & Karweit, 1984, 1985) and focused on how teachers used domain-specific knowledge, and how they organized, sequenced, and presented it in attempts to promote different types of student performance in classroom settings (Berliner et al., 1988). Other studies have centered on efforts to delineate the differences in decision making between "novice" and "expert" teachers in planning for teaching and in instruction (Leinhardt, 1988; Leinhardt & Greeno, 1986). These studies often focus on the teaching acts that differentiate the performances of expert and novice teachers with classification based on student performance on standardized achievement tests, topic-specific tests, or student growth over a period of time. Leinhardt and co-workers have investigated the teaching of fractions, examining the role of teacher decision making, use of scripts, and the role and type of explanations. Good, Grouws, and Ebmeier (1983) examined the role of active teaching by expert teachers and then developed a prototype lesson organization that promoted student growth in mathematics.

Shavelson, Webb, Stasz, and McArthur (1988) provide warnings about the nature of findings from research based on the external conception point of view. First, the findings provide a picture of the existing situation, not a picture of what could be achieved under dramatically changed instruction. Second, the findings reflect the type of performance that was used to separate the teachers into the different categories initially. That is, when teachers were selected as experts on the basis of specific criteria, the results reflect the teaching patterns of instruction related to those criteria. The conduct of the studies and the external conception of the mathematics employed tend to direct the type of research questions asked, and those *not* asked. This research must include teachers with a wide variety of styles if findings generalizable to all teachers or all classrooms are desired.

The second group of researchers adopting the external view espouse a more dynamic view of mathematics, but they focus on adjusting the curriculum to reflect this growth of the discipline and to see how students acquire knowledge of the related content and skills. The underlying focus is, however, still on student mastery of the curriculum or on the application of recent advances in technology or instructional technology to mathematics instruction.

Thorpe (1989), in reviewing the nature of the teaching of algebra, states that "students have needed to learn pretty much the same algebra as did their parents and grandparents. But now something *has* changed. We have new tools" (p. 23). Kaput (1989) took this issue of the changing context for the teaching of algebra and provided a list of research questions concerning the role of linked representations in developing the symbol system of algebra. Thompson (1989) provided additional examples for the development of meanings for topics in numeration and quantity. The work of Wearne and Hiebert (1988) provides another example of such research in mathematics education. Taking the concepts and skills related to fractions as given, Wearne and Hiebert looked at the ways in which students can come to understand and operate with decimal fractions and apply these learnings to situations calling for transfer of understanding to procedural skill. Each of these studies assumes the mathematics as given, but also allows for it to take on new meanings as time passes. The issue at heart is how teacher instruction or student understanding can be improved through research. The focus here is not on the creation of new content, but on the growth of individual knowledge of an existing portion of mathematics.

Internal Conceptions

The remaining three conceptions of mathematics found in mathematics education research focus on mathematics as a personally constructed, or internal, set of knowledge. In the first of these, mathematics is viewed as a process. Knowing mathematics is equated with doing mathematics. Research in this tradition focuses on examining the features of a given context that promotes the "doing." Almost everyone involved in the teaching and learning of mathematics holds that the learning of mathematics is a personal matter in which learners develop their own personalized notions of mathematics as a result of the activities in which they participate. Ernst von Glaserfeld (1987) described this conception of learning and teaching:

> [As we] come to see knowledge and competence as products of the individual's conceptual organization of the individual's experience, the teacher's role will no longer be to dispense "truth" but rather to help and guide the student in the conceptual organization of certain areas of experience. Two things are required for the teacher to do this: on the one hand, an adequate idea of where the student is and, on the other, an adequate idea of the destination. Neither is accessible to direct observation. What the student says and does can be interpreted in terms of a hypothetical model—and this is one area of educational research that every *good* teacher since Socrates has done intuitively. Today, we are a good deal closer to providing the teacher with a set of relatively reliable diagnostic tools.
>
> As for the helping and guiding, good teachers have always found ways and means of doing it because, consciously or unconsciously they realized that, although one can point the way with words and symbols, it is the student who has to do the conceptualizing and the operating. (p. 16)

This emphasis on students doing mathematics is the hallmark of this conceptualization of mathematics. It is the "doing"—the experimenting, abstracting, generalizing, and specializing—that constitutes mathematics, not a transmission of a well-formed communication. This approach to the learning of mathematics is reflected in the writing of Steffe (1988) and Romberg (1988), as well as in many of the emerging activity-oriented preschool and primary programs. This conception seems to be shared by George Polya as expressed in an address to the American Mathematical Society on his views on the learning of mathematics:

> It has been said by many people in many ways that learning should be active, not merely passive or receptive; merely by reading books or

listening to lectures or looking at moving pictures without adding some action of your own mind you can hardly learn anything and certainly you can not learn much.

There is another often expressed (and closely related) opinion: *The best way to learn anything is to discover it by yourself.* Lichtenberg... adds an interesting point: *What you have been obligated to discover by yourself leaves a path in your mind which you can use again when the need arises.* (Polya, 1965, pp. 102–103)

This personal construct approach to mathematics is a strong component of many of the recommendations of the NCTM *Curriculum and Evaluation Standards for School Mathematics* (1989) and has a strong history in mathematics education, including the work of Harold Fawcett (1938), the work of the Progressive Education Association (1938), and the NCTM *Agenda for Action* (1980).

A second personal, or internal, conceptualization of mathematics is based on the description of mathematical activities in terms of psychological models employing cognitive procedures and schemata. Larkin (1989) describes this approach in the following statement:

The central technique of cognitive science is modeling problem-solving behavior in the following way: A problem is considered as a *data* structure that includes whatever information is available about the problem. We then ask what kind of *program* could add information to that data structure to produce a solution to the problem. Because we want to have a model that explains human performance, we require that the model add information to the data structure in orders consistent with the orders in which humans are observed to add information. (p. 120)

This cognitive science approach to the study of mathematics can be found in the works and recommendations of Bransford, et al. (1988); Campione, Brown, and Connell (1988); Carpenter (1988); Chaiklin (1989); Hiebert (1986); Larkin (1989); Marshall (1988); Nesher (1988); Ohlsson (1988); Peterson (1988); Resnick (1987); and Wearne and Hiebert (1988). The diversity of this research adopting the cognitive modeling approach shows the apparent acceptance of it as a model for viewing the structure of mathematics learning. Its basic tenets are the identification of representations for mathematical knowledge, of operations individuals perform on that knowledge, and of the manner in which the human mind stores, transforms, and amalgamates that knowledge.

The third internal conception of mathematics that surfaces in mathematics education research is one that views mathematics knowledge as resulting from social interactions. Here the learning of mathematics is the acquiring of relevant facts, concepts, principles, and skills as a result of social interactions that rely heavily on context. The research describing this view (Bauersfeld, 1980; Bishop, 1985, 1988; Kieren, 1988; Lave, Smith, & Butler, 1988; Schoenfeld, 1988, 1989) focuses on building mathematics knowledge from learning in an apprentice mode, drawing on both the content and the context. Such an approach perhaps heightens the learner's ability to relate the mathematics to its applications and its potential use in problem-solving situations. The distance between the theoretical aspects of the content and the practical distinctions of applications is diminished. In social settings, the measurement of an individual's progress in mathematics is judged on the degree to which he or she has attained the content material transmitted. There is no measure of the cultural information transmitted or the relation of that material to the learner's position in life.

Schoenfeld (1988) argues that the nature of mathematics perceived by students is a result of an intricate interaction of cognitive and social factors existing in the context of schooling. If students are to learn and apply mathematics, they must come to see mathematics as having worth in social settings. Such "sense making" in the learning of mathematics calls for students to participate actively in "doing mathematics" to learn the skills of the discipline. Students must be called upon to participate aggressively in analyzing, conjecturing, structuring, and synthesizing numerical and spatial information in problem settings. These activities must also involve the students in seeing how the results of such activities relate to the solution of problems in the social setting from which the problems originated. Kieren (1988) similarly argues for the careful integration of mathematics learning with the features of the social context in which the mathematics has meaning.

Each of these three conceptions of the development and study of internal models for mathematics education provides important vantage points for research on the learning and teaching of mathematics. The election of one of these philosophies and its use in the design of research strongly influence the nature of the questions investigated, the manner in which relevant data are collected and analyzed, and the importance tied to the conclusions reached. Creators and users of research in mathematics education must pay closer attention to the role such philosophies play in the conduct of that research. To ignore this feature is to misinterpret findings and misapply the outcomes of such studies.

SUMMARY

The survey of the literature shows that conceptions of mathematics fall along an externally-internally developed continuum. Hersh's (1986) comments, along with others (Tymoczko, 1986), indicate that mathematicians behave like constructionalists until challenged. Similar findings may hold for mathematics teachers. The retreat to the external model to discuss their conceptions shows a strong predilection for Platonic views of mathematics. Such conceptions are strongly flavored by dualistic or multiplistic beliefs about mathematics, allowing few teachers to reject an authoritarian teaching style. Even so, the leaders and professional organizations in mathematics education are promoting a conception of mathematics that reflects a decidedly relativistic view of mathematics (Ernest, 1989). Steps to address the gaps between the philosophical bases for current mathematics instruction are important ones that must be addressed in the development and study of mathematics education at all levels.

The emergence of a process view of mathematics embedded in the NCTM *Standards* (1989) and in the works of modern mathematical philosophers (Tymoczko, 1986) presents many new and important challenges. Teacher educators and curriculum developers must become aware of the features and ramifications of the internal and external conceptions, and their ramifications for curricular development and teacher actions.

Further, all involved in applying mathematics education research must recognize the important influences of each conception of mathematics on both the findings cited and on the interpretation and application of such findings. Mathematics educators need to focus on the nature of mathematics in the development of research, curriculum, teacher training, instruction, and assessment as they strive to understand its impact on the learning and teaching of mathematics.

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