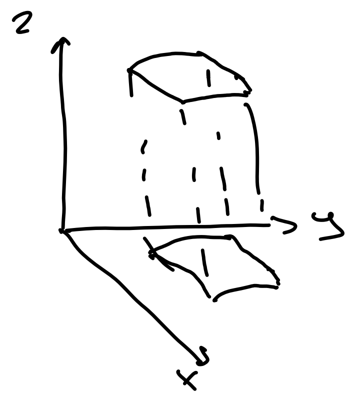
Μια συνάρτηση $z=f(x,y)$ x,y ανεξάρτητες μεταβλητές

& $\forall (x_0,y_0) \in D$ τόπος $z_0 = f(x_0,y_0)$

$z=f(x_1,x_2,\dots,x_n)$, $(x_1,x_2\dots x_n) \in D \subset \mathbb{R}^n$



1/ Μια πολυωνυμική συνάρτηση $f(x,y)$ έχει π.ο. στο $\mathbb{R}^2$

2/ Μια ρητή στο $\mathbb{R}^2$ - όπου μηδεν. ο παρονομαστής

3/ $\sqrt{P(x,y)}$ $P(x,y) \geq 0$

4/ τριγων. $\cos(\varphi(x,y))$ κλπ $\varphi(x,y)$ όπου

5/ $\ln(\varphi(x,y))$ $\varphi(x,y) > 0$

Παράδειγμα.

$f(x,y) = 2x^2y + x + y^3$ $(1,0), (-2,1), (1,\frac{1}{2})$

$f(1,0) = 2\cdot 1^2 0 + 1 + 0^3 = 1$

$f(-2,1) = 2\cdot(-2)^2\cdot 1 + (-2) + 1^3 = 8 - 2 + 1 = 7$

$f(1,1/2) = \cancel{2}\cdot 1^2 \cdot \frac{1}{\cancel{2}} + 1 + \left(\frac{1}{2}\right)^3 = 1 + 1 + \frac{1}{8} = 2 + \frac{1}{8} = \frac{17}{8}$

a/ Καρτεσιανές συντ.

(x,y,z)

(b) Κυλινδρικές συντ.

β) Κυλινδρικές συντ.

$x = r\cos\theta$ $\quad r \geq 0 \quad (r, \theta, z)$

$y = r\sin\theta$ $\quad 0 \leq \theta \leq 2\pi$

$z = z$

γ) Πολικές συντ.

$x = r\sin\varphi\cos\theta$ $\quad r \geq 0$

$y = r\sin\varphi\sin\theta$ $\quad 0 \leq \theta \leq 2\pi \quad (r, \theta, \varphi)$

$z = r\cos\varphi$ $\quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

Όριο $z = f(x, y)$ $\quad x \to x_0$, $y \to y_0$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = z_0 = l$$

Όταν $\exists\, \varepsilon > 0$ κ' $\delta > 0$ όπου δ εξαρτάται από το ε : $|f(x,y) - l| < \varepsilon$

$0 < |x - x_0| < \delta \quad , \quad 0 < |y - y_0| < \delta$

$$\lim_{x \to x_0} \left(\lim_{y \to y_0} f(x, y) \right) \qquad \lim_{y \to y_0} \left(\lim_{x \to x_0} f(x, y) \right)$$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = l$$

Άσκηση

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right) = 1$$

$$\lim_{\substack{x \to 0 \\ y \to 2}} \frac{\sin(xy)}{x}$$

$$\lim_{x \to 0} \left(\lim_{y \to 2} \frac{\sin(xy)}{x} \right) = \lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{\substack{x \to 0 \\ 2x \to 0}} 2 \cdot \frac{\sin(2x)}{2x}$$

$$\lim_{x\to 0}\left(\lim_{y\to 2} \frac{\quad}{x}\right) = \lim_{x\to 0} \frac{\quad}{x} \qquad \begin{matrix} x\to 0 \\ 2x\to 0 \end{matrix} \qquad 2x$$

$$u = 2x$$

$$= 2 \cdot \lim_{u\to 0} \frac{\sin(u)}{u} = 2\cdot 1 = 2$$

$$\lim_{y\to 2}\left(\lim_{x\to 0} \frac{\sin(xy)}{x}\right) = \lim_{y\to 2}\left(\lim_{\substack{x\to 0\\ xy\to 0}} y \cdot \frac{\sin(xy)}{xy}\right) =$$

$$= \lim_{y\to 2} y \left(\lim_{xy\to 0} \frac{\sin(xy)}{xy}\right) = 2 \cdot 1 = 2$$

$$f(x,y) = \frac{\sin(xy)}{x}$$

$$|f(x,y)| = \left|\frac{\sin(xy)}{x}\right| = \left| y \frac{\sin(xy)}{yx}\right| =$$

$$|y| \left|\frac{\sin(xy)}{xy}\right| \le |y| \cdot 1 \qquad \le 1$$

$$\lim_{y\to 2} |y| = |2| = 2$$

$$\lim |f(x,y)| = \lim f(x,y) = 2$$

Άσκηση

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2}$$

$$\lim_{x\to 0}\left(\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{x^2}{x^2}\right) = \lim_{x\to 0} \frac{1}{1} = 1$$

$$\neq$$

$$\lim\left(\lim \frac{x^2-y^2}{x^2+y^2}\right) = \lim\left(\frac{-y^2}{y^2}\right) = -1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}\right)=\lim_{y\to 0}\left(\frac{-y^2}{y^2}\right)=-1$$

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2-y^2}{x^2+y^2}\qquad \begin{aligned} x&=r\cos\theta=\rho\cos\theta\\ y&=r\sin\theta=\rho\sin\theta\end{aligned}$$

$$\lim_{\substack{r\to 0\\ \text{ή } \rho\to 0}}\frac{r^2\cos^2\theta-r^2\sin^2\theta}{r^2\cos^2\theta+r^2\sin^2\theta}=\lim_{r\to 0}\frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}=$$

$$=\lim_{r\to 0}\frac{\cos^2\theta-\sin^2\theta}{1}$$

εξαρτάται από την γωνία θ άρα $\nexists$ όριο

Ασκηση

$$z=f(x,y)=\frac{3y^2}{x^2+y^2}\qquad \exists;\ \lim\ (x,y)\to(0,0)$$

$$x=\rho\cos\theta,\quad y=\rho\sin\theta$$

$$\lim_{\substack{x\to 0\\ y\to 0}}f(x,y)=\lim_{\rho\to 0}\frac{3\rho^2\sin^2\theta}{\rho^2\cos^2\theta+\rho^2\sin^2\theta}=$$

$$=\lim_{\rho\to 0}\frac{3\sin^2\theta}{1}$$

εξαρτάται από θ άρα $\nexists$ lim

Ασκηση $\exists$ lim

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right)\right)$$

$$\left|x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right)\right|\le\left|x\sin\left(\frac{1}{y}\right)\right|+\left|y\sin\left(\frac{1}{x}\right)\right|$$

$$\le|x|\left|\sin\left(\frac{1}{y}\right)\right|+|y|\left|\sin\left(\frac{1}{x}\right)\right|\le$$

$$\left|\sin\left(\frac{1}{x}\right)\right|\le 1$$

$\leq |x||\sin(\frac{1}{y})| + |y||\sin(\frac{1}{x})|$

$|\sin(\frac{1}{y})|$ κ' $|\sin(\frac{1}{x})| \leq 1$

$\leq |x| + |y|$

$\lim_{\substack{x\to 0\\ y\to 0}} (|x|+|y|) = 0$ $\qquad \lim_{\substack{x\to 0\\ y\to 0}} \ldots = 0$

Άσκηση

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3 y}{x^2+y^2}$ $\qquad x=\rho\cos\theta$ ή $x = r\cos\theta$

$y = \rho\sin\theta$ ή $y = r\sin\theta$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3 y}{x^2+y^2} = \lim_{\rho\to 0} \frac{\rho^{\cancel{3}}\cos^3\theta\,\rho\sin\theta}{\cancel{\rho^2}(\cos^2\theta+\sin^2\theta)} =$

$= \lim_{\rho\to 0} \rho^2\cos^3\theta\sin\theta = 0$

$\left|\frac{x^3 y}{x^2+y^2}\right| \leq \ldots$ $\qquad = 0$

$x^2+y^2-2xy \geq 0 \Leftrightarrow x^2+y^2 \geq 2|xy|$

$\frac{1}{x^2+y^2} \leq \frac{1}{2|x||y|} \Leftrightarrow \frac{|x^3 y|}{x^2+y^2} \leq \frac{|x^3 y|}{2|x||y|} \leq \frac{|x^3||\cancel{y}|}{2|x|\cancel{|y|}} \leq$

$\leq \frac{|x|^2}{2}$

$\lim_{x\to 0} \frac{|x|^2}{2} = 0$

Άσκηση

Άσκηση

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2+3y^2}{x^2+y^2} \qquad \begin{matrix} x=\rho\cos\varphi \\ y=\rho\sin\varphi \end{matrix}$$

$$\lim_{\rho\to 0} \frac{\rho^2\cos^2\varphi+3\rho^2\sin^2\varphi}{\rho^2\cos^2\varphi+\rho^2\sin^2\varphi} = \lim_{\rho\to 0} \frac{\rho^2(\cos^2\varphi+3\sin^2\varphi)}{\rho^2(\underbrace{\cos^2\varphi+\sin^2\varphi}_{1})} =$$

$$\lim_{\rho\to 0}\left(\cos^2\varphi+3\sin^2\varphi\right) = \lim_{\rho\to 0} \left(1-\sin^2\varphi\right)+3\sin^2\varphi =$$

$$= \lim_{\rho\to 0} 1+2\sin^2\varphi \qquad \nexists \text{ το όριο}$$

Άσκηση

$$\lim_{\substack{x\to 1\\ y\to 2}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \lim_{\substack{u\to 0\\ v\to 0}} \frac{uv}{u^2+v^2}$$

$x-1\to 0 \quad u=x-1 \qquad u=\rho\cos\varphi$

$y-2\to 0 \quad v=y-2 \qquad v=\rho\sin\varphi$

$$= \lim_{\rho\to 0} \frac{\rho\cos\varphi\cdot\rho\sin\varphi}{\rho^2\cos^2\varphi+\rho^2\sin^2\varphi} = \lim_{\rho\to 0} \frac{\cancel{\rho^2}\cos\varphi\sin\varphi}{\cancel{\rho^2}(\cos^2\varphi+\sin^2\varphi)} =$$

$$= \cos\varphi\sin\varphi. \qquad \nexists \lim$$

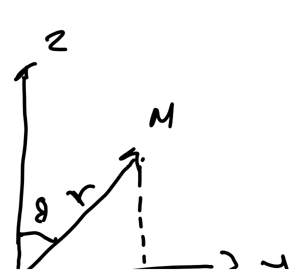
Άσκηση

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}} \frac{2x-y+3z}{x+2y-z}$$

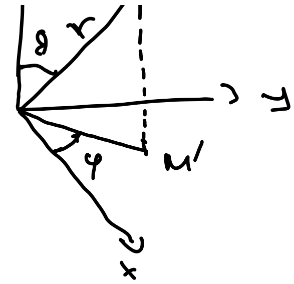
$x=r\sin\theta\cos\varphi$

$y=r\sin\theta\sin\varphi$

$z=r\cos\theta$



$$\lim_{\substack{y\to 0\\ z\to 0}}$$



$$\lim_{r\to 0}\frac{2r\sin\theta\cos\varphi - r\sin\theta\sin\varphi + 3r\cos\theta}{r\sin\theta\cos\varphi + 2r\sin\theta\sin\varphi - r\cos\theta} =$$

$$\lim_{r\to 0}\frac{\cancel{r}(2\sin\theta\cos\varphi - \sin\theta\sin\varphi + 3\cos\theta)}{\cancel{r}(\sin\theta\cos\varphi + 2\sin\theta\sin\varphi - \cos\theta)} =$$

$\nexists \lim$

Άσκηση

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}}\frac{3y^2 z^2}{x^2+y^2+z^2}$$

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$

$$\lim_{r\to 0}\frac{3\cdot r^2\sin^2\theta\sin^2\varphi\cdot r^2\cos^2\theta}{r^2\sin^2\theta\cos^2\varphi + r^2\sin^2\theta\sin^2\varphi + r^2\cos^2\theta} =$$

$$=\lim_{r\to 0} 3r^4\frac{\sin^2\theta\cos^2\varphi\sin^2\varphi}{r^2(\sin^2\theta(\underbrace{\cos^2\varphi+\sin^2\varphi}_{1}) + \cos^2\theta)} =$$

$$=\lim_{r\to 0} 3r^2\frac{\sin^2\theta\cos^2\varphi\sin^2\varphi}{\underbrace{\sin^2\theta+\cos^2\theta}_{1}} =$$

$$=\lim_{r\to 0} 3r^2\sin^2\theta\cos^2\varphi\sin^2\varphi = 0$$

$$f(x,y,z) = \frac{3y^2z^2}{x^2+y^2+z^2}$$

$x^2+y^2+z^2 \geq z^2 \qquad \frac{1}{x^2+y^2+z^2} \leq \frac{1}{z^2} \Rightarrow$

$\Rightarrow \frac{z^2}{x^2+y^2+z^2} \leq 1$

$$\left| \frac{3y^2 z^2}{x^2+y^2+z^2} \right| \leq |3y^2|$$

$$\lim_{\substack{x \to 0 \\ y \to 0 \\ z \to 0}} 3y^2 = 0$$

Άρα $\exists$ lim $= 0$

Μερική Παράγωγοι

$z = f(x,y)$ σε σύνολο D $(x_0, y_0) \in D$

$f(x, y_0)$

$$\frac{\partial (x_0, y_0)}{\partial x} = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0, y_0)}{x - x_0}$$

$$\frac{\partial (x_0, y_0)}{\partial x} \text{ ή } \frac{\partial}{\partial x} f(x_0, y_0) \text{ ή } \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} \text{ ή } f'(x_0, y_0)$$

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}, \quad (x,y) \in D^* \subset D$$

$x - x_0 = h$

Αντίστοιχα ως προς y

Παράδειγμα. Να βρω. μερικές παραγώγους ως προς x, y

$2xy^3$

σταθερά

$$\frac{\partial}{\partial x}(2xy^3) = 2y^3 \frac{\partial}{\partial x}(x) = 2y^3 (x)' = 2y^3$$

σταθερά

dx

$$\frac{\partial}{\partial y}\left(2\overset{\text{σταθερά}}{x}y^3\right) = 2x\frac{\partial}{\partial y}(y^3) = 2\cdot 3x\, y^2 = 6xy^2$$

Άσκηση $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$

$f(x,y) = x + y^2x^2$

$$\frac{\partial}{\partial x}(x+y^2x^2) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y^2x^2) =$$

$$= 1 + y^2\frac{\partial}{\partial x}(x^2) = 1 + 2y^2x$$

$$\frac{\partial}{\partial y}(x+y^2x^2) = \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(y^2x^2) =$$

$$= 0 + 2yx^2$$

Άσκηση $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$,

$x^4y^2 + 2x + 2y^2 + 4$, $\sin(xy)$, $\cos(x^2+y^2)$

$$\frac{\partial}{\partial x}(x^4y^2 + 2x + 2y^2 + 4) = 4x^3y^2 + 2$$

$$\frac{\partial}{\partial y}(x^4y^2 + 2x + 2y^2 + 4) = 2x^4y + 4y$$

$$\frac{\partial}{\partial x}(\sin(xy)) = \cos(xy)\cdot(xy)\Big|_x = y\cos(xy)$$

$$\frac{\partial}{\partial y}(\sin(xy)) = \cos(xy)\cdot(xy)\Big|_y = x\cos(xy)$$

$$\frac{\partial}{\partial x}(\cos(x^2+y^2)) = -\sin(x^2+y^2)\cdot(x^2+y^2)\Big|_x$$

$$\frac{\partial}{\partial x}\left(\cos(x^2+y^2)\right) = -\sin(x^2+y^2)\cdot(x^2+y^2)'_x$$
$$= -\sin(x^2+y^2)\cdot 2x$$
$$= -2x\sin(x^2+y^2)$$

$$\frac{\partial}{\partial y}\left(\cos(x^2+y^2)\right) = -2y\sin(x^2+y^2)$$

Άσκηση $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ e^{x^2+2xy}

$$\frac{\partial}{\partial x}\left(e^{x^2+2xy}\right) = e^{x^2+2xy}\cdot\frac{\partial}{\partial x}(x^2+2xy) =$$
$$= e^{x^2+2xy}(2x+2y) = 2(x+y)\cdot e^{x^2+2xy}$$

$$\frac{\partial}{\partial y}\left(e^{x^2+2xy}\right) = e^{x^2+2xy}\cdot\frac{\partial}{\partial y}(x^2+2xy) =$$
$$= e^{x^2+2xy}(0+2x) = 2x\,e^{x^2+2xy}$$

Άσκηση $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

$$f(x,y,z) = xyz^2\sin(x+y)$$
$$g(x,y,z) = xz\ln(y^2+xz)$$
$$\varphi(x,y,z) = \frac{xz^2}{x+y+z}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\cdot g'(x)}{g^2(x)}$$

$$f(x,y,z) = xyz^2\sin(x+y)$$
$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xyz^2\sin(x+y)\right) = \frac{\partial}{\partial x}(xyz^2)\sin(x+y) + (xyz^2)\frac{\partial}{\partial x}(\sin(x+y))$$
$$yz^2\sin(x+y) + (xyz^2)\cdot\cos(x+y)\frac{\partial}{\partial x}(x+y) =$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xyz^2\sin(x+y)\right) \frac{\partial}{\partial x}$$

$$= yz^2\sin(x+y) + (xyz^2)\cdot\cos(x+y)\frac{\partial}{\partial x}(x+y) =$$

$$= yz^2\sin(x+y) + (xyz^2)\cdot\cos(x+y)\cdot 1$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xyz^2\sin(x+y)\right) = \frac{\partial}{\partial y}(xyz^2)\sin(x+y) + (xyz^2)\frac{\partial}{\partial y}\sin(x+y) =$$

$$= xz^2\sin(x+y) + (xyz^2)\cdot\cos(x+y)\frac{\partial}{\partial y}(x+y) =$$

$$= yz^2\sin(x+y) + (xyz^2)\cos(x+y)\cdot 1$$

$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(xyz^2\sin(x+y)\right) = xy\sin(x+y)\frac{\partial}{\partial z}(z^2) = 2xyz\sin(x+y)$$

$$g(x,y,z) = xz\ln(\underbrace{y^2+x\cdot z}_{g(x)}) \qquad (f(x)\cdot g(x))' = f'(x)g(x) + f(x)\cdot g'(x)$$

$$g_x = \frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(\underbrace{xz}_{f(x)}\underbrace{\ln(y^2+x\cdot z)}_{g(x)}\right) = z\cdot\ln(y^2+x\cdot z)\frac{\partial}{\partial x}(x) + xz\frac{\partial}{\partial x}\left(\ln(y^2+x\cdot z)\right)$$

$$= z\cdot\ln(y^2+x\cdot z) + x\cdot z\,\frac{1}{y^2+x\cdot z}\,\frac{\partial}{\partial x}(y^2+xz) =$$

$$= z\ln(y^2+x\cdot z) + x\cdot z\,\frac{1}{y^2+x\cdot z}\,z = z\ln(y^2+x\cdot z) + \frac{x\cdot z^2}{y^2+x\cdot z}$$

$$g_y = \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(x\cdot z\ln(y^2+x\cdot z)\right) = x\cdot z\frac{\partial}{\partial y}\left(\ln(y^2+x\cdot z)\right) =$$

$$= x\cdot z\cdot\frac{1}{y^2+x\cdot z}\,\frac{\partial}{\partial y}(y^2+x\cdot z) =$$

$$= x\cdot z\cdot\frac{1}{y^2+x\cdot z}\,2y = \frac{2xyz}{y^2+x\cdot z} \qquad (f(x)\,g(x))' = f'(x)g(x) + f(x)\,g'(x)$$

$$g_z = \frac{\partial g}{\partial z} = \frac{\partial}{\partial z}\left(\underbrace{x\cdot z}_{f(x)}\cdot\underbrace{\ln(y^2+x\cdot z)}_{g(x)}\right) =$$

$$= \frac{\partial}{\partial z}(xz)\cdot\ln(y^2+xz) + (x\cdot z)\frac{\partial}{\partial z}\left(\ln(y^2+x\cdot z)\right) =$$

$$= x\ln(y^2+x\cdot z) + (x\cdot z)\frac{1}{y^2+x\cdot z}\,\frac{\partial}{\partial z}(y^2+x\cdot z) =$$

$$= x\ln(y^2+x\cdot z) + (xz)\,\frac{1}{y^2+x\cdot z}\,x =$$

$$= x\ln(y^2+x\cdot z) + \frac{x^2z}{y^2+x\cdot z}$$

$$x\cdot z^2 \qquad \left(\frac{f(x)}{\ }\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\ }$$

$$\varphi(x,y,z) = \frac{x\cdot z^2}{x+y+z} \qquad \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x\cdot z^2}{x+y+z}\right) = \frac{\frac{\partial}{\partial x}(x\cdot z^2)\cdot(x+y+z) - (x\cdot z^2)\frac{\partial}{\partial x}(x+y+z)}{(x+y+z)^2} =$$

$$= \frac{z^2\cdot(x+y+z) - (x\cdot z^2)\cdot 1}{(x+y+z)^2} = \frac{z^2(y+z)}{(x+y+z)^2}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x\cdot z^2}{x+y+z}\right) = \frac{\frac{\partial}{\partial y}(x\cdot z^2)(x+y+z) - (x\cdot z^2)\cdot\frac{\partial}{\partial x}(x+y+z)}{(x+y+z)^2} =$$

$$= \frac{0\cdot(x+y+z) - (x\cdot z^2)\cdot 1}{(x+y+z)^2} = \frac{-xz^2}{(x+y+z)^2}$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x\cdot z^2}{x+y+z}\right) = \frac{\frac{\partial}{\partial z}(x\cdot z^2)\cdot(x+y+z) - (x\cdot z^2)\frac{\partial}{\partial z}(x+y+z)}{(x+y+z)^2} =$$

$$= \frac{2zx(x+y+z) - x\cdot z^2\cdot 1}{(x+y+z)^2} = \frac{x\cdot z(2x+2y+z)}{(x+y+z)^2}$$

Μερικές παράγωγοι διανυσματικής συνάρτησης·

$$\bar r(x,y) = (f(x,y), g(x,y))$$

$$\frac{\partial \bar r}{\partial x} = \left(\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x}\right) \quad \text{κ'} \quad \frac{\partial \bar r}{\partial y} = \left(\frac{\partial f}{\partial y}, \frac{\partial g}{\partial y}\right)$$

Μερικές παράγωγοι ανώτερης τάξης

2ης τάξη

$f(x,y)$

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \text{κ'} \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Παράδειγμα

$$z = f(x,y) = \ln(x^2+y^2) - x^3y^2$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\ln(x^2+y^2) - x^3y^2\right) =$$

$$= \frac{\partial}{\partial x}\left(\ln(x^2+y^2)\right) - \frac{\partial}{\partial x}\left(x^3y^2\right) =$$

$$= \frac{1}{x^2+y^2}\,\frac{\partial}{\partial x}(x^2+y^2) - 3x^2y^2$$

$$= \frac{1}{x^2+y^2}\,2x = \frac{2x}{x^2+y^2} - 3x^2y^2$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\ln(x^2+y^2) - x^3y^2\right) = \frac{\partial}{\partial y}\left(\ln(x^2+y^2)\right) - \frac{\partial}{\partial y}(x^3y^2)$$

$$= \frac{1}{x^2+y^2}\,\frac{\partial}{\partial y}(x^2+y^2) - 2x^3y = \frac{2y}{x^2+y^2} - 2x^3y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2} - 3x^2y^2\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) - \frac{\partial}{\partial x}(3x^2y^2)$$

$$= \frac{\frac{\partial}{\partial x}(2x)\cdot(x^2+y^2) - (2x)\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} - 6xy^2 =$$

$$-6xy^2 \qquad -6xy^2$$

$$= \frac{2\cdot(x^2+y^2) - 2x\cdot 2x}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} - 6xy^2$$

$$= \frac{\ldots}{(x^2+y^2)^2} - \frac{\ldots}{(x^2+y^2)^2} \qquad (x^2+y^2)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2} - 2x^3 y\right) =$$

$$= \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(2x^3y\right) =$$

$$= \frac{\frac{\partial}{\partial y}(2y)\cdot(x^2+y^2) - (2y)\cdot\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} - 2x^3 =$$

$$= \frac{2(x^2+y^2) - 2y\cdot 2y}{(x^2+y^2)^2} - 2x^3 = \frac{2x^2-2y^2}{(x^2+y^2)^2} - 2x^3$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2} - 2x^3y\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right) - \frac{\partial}{\partial x}(2x^3y) =$$

$$= \frac{\frac{\partial}{\partial x}(2y)\cdot(x^2+y^2) - 2y\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} - 6x^2y$$

$$= \frac{-2y\cdot 2x}{(x^2+y^2)^2} - 6x^2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2} - 3x^2y^2\right) =$$

$$= \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(3x^2y^2\right) =$$

$$= \frac{\frac{\partial}{\partial y}(2x)\cdot(x^2+y^2) - (2x)\cdot\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} - 6x^2y$$

$$= \frac{-2x\cdot 2y}{(x^2+y^2)^2} - 6x^2y$$

$1/\ f(x,y) = e^{-5x}\cos y$ μερ. παραγ. $1^{ης}$ κ' $2^{ης}$ τάξης

$1/\ f(x,y) = e^{-5x}\cos y$ μερ. παραγ. 1ης κ' 2ης τάξης

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{-5x}\cos y\right) = e^{-5x}\frac{\partial}{\partial x}(-5x)\cos y = -5\cdot e^{-5x}\cdot\cos y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{-5x}\cdot\cos y\right) = e^{-5x}\cdot\frac{\partial}{\partial y}(\cos y) = -e^{-5x}\cdot\sin y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(-5\cdot e^{-5x}\cos y\right) = -5\cdot e^{-5x}\frac{\partial}{\partial x}(-5x)\cdot\cos y =$$
$$= -5\cdot e^{-5x}\cdot(-5)\cdot\cos y = 25\cdot e^{-5x}\cdot\cos y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-5\cdot e^{-5x}\cdot\sin y\right) = -5\cdot e^{-5x}\frac{\partial}{\partial y}(\sin y) =$$
$$= -5\cdot e^{-5x}\cos y$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-5\cdot e^{-5x}\sin y\right) = -5\cdot e^{-5x}\frac{\partial}{\partial x}(-5x)\cdot\sin y =$$
$$= 25e^{-5x}\cdot\sin y$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(-5\cdot e^{-5x}\cos y\right) = -5\cdot e^{-5x}\frac{\partial}{\partial y}(\cos y) =$$
$$= -5\cdot e^{-5x}\cdot(-\sin y) = 5\cdot e^{-5x}\sin y$$

<u>Ασκήσεις</u>

$g(x,y) = x^2y^3 + 2x^3\ln y$

$h(x,y) = \dfrac{x}{y} + \tan(xy)$ εφαπ.

$$(\tan(x))' = \frac{1}{\cos^2(x)}$$

$$(\tan(f(x)))' = \frac{1}{\cos^2(f(x))}\cdot(f(x))'$$

$$g(x,y) = x^2y^3 + 2x^3\ln y$$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(2x^3\ln y) =$$
$$= 2xy^3 + 6x^2\ln y$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(x^2y^3\right) + \frac{\partial}{\partial y}\left(2x^3\ln y\right) =$$

$$= 3x^2y^2 + 2x^3\frac{1}{y}$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xy^3 + 6x^2\ln y\right) =$$

$$= \frac{\partial}{\partial x}\left(2xy^3\right) + \frac{\partial}{\partial x}\left(6x^2\ln y\right) =$$

$$= 2y^3 + 12x\ln y$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}\left(3x^2y^2 + 2x^3\frac{1}{y}\right) =$$

$$= \frac{\partial}{\partial y}\left(3x^2y^2\right) + \frac{\partial}{\partial y}\left(2x^3\frac{1}{y}\right) =$$

$$\left(\frac{1}{y}\right)' = -\frac{1}{y^2}$$

$$= 6x^2y - 2x^3\frac{1}{y^2}$$

$$\frac{\partial^2 g}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2y^2 + 2x^3\frac{1}{y}\right) =$$

$$= \frac{\partial}{\partial x}\left(3x^2y^2\right) + \frac{\partial}{\partial x}\left(2x^3\frac{1}{y}\right) =$$

$$= 6xy^2 + 6x^2\frac{1}{y}$$

$$\frac{\partial^2 g}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy^3 + 6x^2\ln y\right) =$$

$$= \frac{\partial}{\partial y}\left(2xy^3\right) + \frac{\partial}{\partial y}\left(6x^2\ln y\right) =$$

$$= 6xy^2 + 6x^2\frac{1}{y}$$

$= 6xy + 6x\ y$

$h(x,y) = \frac{x}{y} + \tan(xy)$

$(\tan(x))' = \frac{1}{\cos^2 x}$

$\tan(f(x))' = \frac{1}{\cos^2 f(x)} \cdot f'(x)$

$$\frac{\partial}{\partial x} h = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{\partial}{\partial x}\left(\tan(xy)\right)$$

$$= \frac{1}{y} + \frac{1}{\cos^2(xy)} \frac{\partial}{\partial x}(xy)$$

$$= \frac{1}{y} + \frac{1}{\cos^2(xy)}\ y$$

$$\frac{\partial}{\partial y} h = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{\partial}{\partial y}\left(\tan(xy)\right)$$

$$= -\frac{x}{y^2} + \frac{1}{\cos^2(xy)} \cdot \frac{\partial}{\partial y}(xy)$$

$$= -\frac{x}{y^2} + \frac{1}{\cos^2(xy)}\ x$$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

$\left(\frac{1}{g(x)}\right)' = -\frac{1}{g^2(x)}\ g'(x)$

$\left(\frac{1}{y}\right)' = -\frac{1}{y^2}$

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{y} + \frac{1}{\cos^2(xy)}\ y\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) + \frac{\partial}{\partial x}\left(\frac{1}{\cos^2(xy)} \cdot y\right)$$

$$= 0 + y \cdot \frac{\partial}{\partial x}\left(\frac{1}{\cos^2(xy)}\right)$$

$$= \frac{2\cos(xy)\sin(xy) \cdot y^2}{\cos^4(xy)}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\cos^2(xy)}\right) =$$

$$= -\frac{1}{\cos^4(xy)} \cdot \frac{\partial}{\partial x}\left(\cos^2(xy)\right)$$

$$= -\frac{1}{\cos^4(xy)} \cdot 2\cdot\cos(xy) \cdot \frac{\partial}{\partial x}(\cos(xy))$$

$$= +\frac{1}{\cos^4(xy)} \cdot 2\cos(xy)\sin(xy)\frac{\partial}{\partial x}(xy)$$

$$= \frac{1}{\cos^4(xy)} \cdot 2\cos(xy)\sin(xy) \cdot y$$

$$\frac{\partial^2 h}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{x}{y^2} + \frac{1}{\cos^2(xy)}\ x\right) =$$

$$= \frac{\partial}{\partial y}\left(-\frac{x}{y^2}\right) + x\frac{\partial}{\partial y}\left(\frac{1}{\cos^2(xy)}\right)$$

$$= \frac{2xy}{y^4} + \frac{2\cos(xy)\sin(xy)\ x^2}{\cos^4(xy)}$$

$$\left(-\frac{x}{y^2}\right)' = -\frac{x' y^2 - x(y^2)'}{y^4} = +\frac{2xy}{y^4}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{\cos^2(xy)}\right) = \frac{2\cos(xy)\cdot\sin(xy)\ x}{\cos^4(xy)}$$

$$= \frac{2xy}{y^4} + \frac{2\cos(xy)\sin(xy)\,x^2}{\cos^4(xy)} \qquad \frac{\partial}{\partial y}\left(\frac{1}{\cos^2(xy)}\right) = \frac{2\cos(xy)\cdot\sin(xy)\,x}{\cos^4(xy)}$$

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{x}{y^2} + \frac{1}{\cos^2(xy)}\cdot x\right) =$$

$$= \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right) + \frac{\partial}{\partial x}\left(\frac{1}{\cos^2(xy)}x\right) + \frac{\partial}{\partial x}(x)\cdot\frac{1}{\cos^2(xy)}$$

$$= -\frac{1}{y^2} \quad \frac{2\cos(xy)\sin(xy)y\,x}{\cos^4(xy)} + \frac{1}{\cos^2(xy)} \qquad \frac{f(x)'g(x) - f(x)g'(x)}{g^2(x)}$$

$$\frac{\partial^2 h}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y} + \frac{1}{\cos^2(xy)}\,y\right) =$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{y}\right) + \frac{\partial}{\partial y}\cdot\left(\frac{1}{\cos^2 xy)}\right)y + \partial(y)\cdot\frac{1}{\cos^2(xy)}$$

$$= -\frac{1}{y^2} + \frac{2\cos(xy)\sin(xy)\,x\,y}{\cos^4(xy)} + \frac{1}{\cos^2(xy)}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\cos^2(xy)}\right) = \frac{\frac{\partial}{\partial y}(y)\cos^2(xy) - y\frac{\partial}{\partial y}(\cos^2(xy))}{\cos^2(xy)} =$$

$$= \frac{\cancel{\cos^2}(xy)}{\cancel{\cos^4}(xy)} \quad -y\cdot\frac{\partial}{\partial y}\left(\cos^2(xy)\right) \Big/ \cos^4(xy)$$

$$= \frac{1}{\cos^2(xy)} \quad \frac{-y\cdot 2\cos(xy)\cdot\frac{\partial}{\partial y}\cos(xy)}{\cos^4(xy)} =$$

$$= \frac{1}{\cos^2(xy)} \quad \frac{+y\,2\cos(xy)\sin(xy)\cdot\frac{\partial}{\partial y}(xy)}{\cos^4(xy)} =$$

$$= \frac{1}{\cos^2(\ )} + \frac{2\cos(xy)\sin(xy)\,xy}{\ }$$

$$= \frac{1}{\cos^2(xy)} + \frac{2\cos(xy)\sin(xy)\,xy}{\cos^4(xy)}$$

Μερικές παράγωγοι σύνθετης συνάρτησης

$z = f(x,y)$ όπου $x = x(t)$, $y = y(t)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$$

Μερικές παράγωγοι

$f = f(x,y)$, $x = x(t,u)$ $y = y(t,u)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = f_x x_t + f_y y_t$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = f_x x_u + f_y y_u$$

$f(x,y)$ $x = x(t)$ $y = y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{d^2f}{dt^2} = \frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{dx}{dt} + \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2} + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dt} + \frac{\partial f}{\partial y}\cdot\frac{d^2y}{dt^2}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}\cdot\frac{dx}{dt} + \frac{\partial^2 f}{\partial x\partial y}\frac{dy}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\partial x}\cdot\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{dy}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x}\cdot\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{dy}{dt}$$

$$\frac{d^2 f}{dt^2} = \left(\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 f}{\partial x \partial y}\frac{dy}{dt}\right)\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} +$$

$$+\left(\frac{\partial^2 f}{\partial y\partial x}\frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt}\right)\cdot\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} =$$

$$= \underbrace{f_{xx}\left(\frac{dx}{dt}\right)^2 + 2f_{xy}\frac{dx}{dt}\frac{dy}{dt} + f_{yy}\left(\frac{dy}{dt}\right)^2} + f_x\frac{d^2x}{dt^2} + f_y\frac{d^2y}{dt^2}$$

$$= \left(\left(f_x\frac{dx}{dt}\right) + \left(f_y\frac{dy}{dt}\right)\right)^2 + f_x\frac{d^2x}{dt^2} + f_y\frac{d^2y}{dt^2}$$

Άσκηση

$\frac{d^2f}{dt^2}$ $\quad f = x^3 + y^2\cdot z$, $x = t^2$, $y = \sin t$, $z = \cos t$

α/ $f = t^6 + \sin^2 t\cdot\cos t$,

$$\frac{df}{dt} = 6t^5 + 2\sin t\cdot(\sin t)'\cdot\cos t + \sin^2 t(\cos t)'$$
$$= 6t^5 + 2\sin t\cos^2 t - \sin^2 t\sin t$$
$$= 6t^5 + 2\sin t\cos^2 t - \sin^3 t$$

$$\frac{d^2f}{dt^2} = 30t^4 + 2(\sin t)'\cos^2 t + 2\sin t(\cos^2 t)' - (\sin^3 t)'$$
$$= 30t^4 + 2\cos t\cdot\cos^2 t + 4\sin t\cdot\cos t(\cos t)' - 3\sin^2 t(\sin t)'$$
$$= 30t^4 + 2\cos^3 t - 4\sin^2 t\cos t - 3\sin^2 t\cos t$$
$$= 30t^4 + 2\cos^3 t - 7\sin^2 t\cos t$$

$x(t) = t^2$
$y(t) = \sin t$
$z(t) = \cos t$
$f = x^3 + y^2 z$

β/ $$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= 3x^2\cdot 2t + 2yz\cdot\cos t - y^2\sin t$$
$$= 6x^2 t + 2yz\cos t - y^2\sin t$$

$$\frac{d^2f}{dt^2} = \left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt}\right)^{(2)} + f_x\frac{d^2x}{dt^2} + f_y\frac{d^2y}{dt^2}$$

$$\ldots \frac{\partial f}{\partial z}\frac{dz}{dt}\Big)^{(2)} \ldots \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial f}{\partial z}\cdot\frac{d^2z}{dt^2}$$

$$\frac{d^2f}{dt^2} = \left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right) \ldots$$

$$\frac{d^2 f}{dt^2} = \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\right)^{(2)} + \frac{\partial f}{\partial x}\cdot\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial f}{\partial z}\cdot\frac{d^2z}{dt^2}$$

$$= f_{xx}\left(\frac{dx}{dt}\right)^2 + f_{yy}\left(\frac{dy}{dt}\right)^2 + f_{zz}\left(\frac{dz}{dt}\right)^2 + 2f_{xy}\frac{dx}{dt}\frac{dy}{dt} + 2f_{xz}\frac{dx}{dt}\frac{dz}{dt} + 2f_{yz}\frac{dy}{dt}\frac{dz}{dt}$$

$$+ f_x \frac{d^2x}{dt^2} + f_y \frac{d^2y}{dt^2} + f_z \frac{d^2z}{dt^2} =$$

Παράγωγος ορίζουσας

Αν $f_t = \begin{vmatrix} f_{11}(t) & f_{12}(t) & \ldots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \ldots & f_{2n}(t) \\ \vdots & & & \\ f_{n1}(t) & f_{n2}(t) & \ldots & f_{nn}(t) \end{vmatrix}$

$$\frac{df(t)}{dt} = \begin{vmatrix} \frac{df_{11}(t)}{dt} & \frac{df_{12}(t)}{dt} & \ldots & \frac{df_{1n}(t)}{dt} \\ f_{21}(t) & f_{22}(t) & \ldots & f_{2n}(t) \\ \vdots & & & \\ f_{n1}(t) & f_{n2}(t) & \ldots & f_{nn}(t) \end{vmatrix} +$$

$$+ \begin{vmatrix} f_{11}(t) & f_{12}(t) & \ldots & f_{1n}(t) \\ \frac{df_{21}(t)}{dt} & \frac{df_{22}(t)}{dt} & \ldots & \frac{df_{2n}(t)}{dt} \\ \vdots & & & \\ f_{n1}(t) & f_{n2}(t) & \ldots & f_{nn}(t) \end{vmatrix} + \ldots +$$

$$+ \begin{vmatrix} f_{11}(t) & f_{12}(t) & \ldots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \ldots & f_{2n}(t) \\ \vdots & & & \\ \frac{df_{n1}(t)}{dt} & \frac{df_{n2}(t)}{dt} & \ldots & \frac{df_{nn}(t)}{dt} \end{vmatrix}$$

Ολικό διαφορικό

... εξαρτημένη μεταβλητή (n,x) z,

Ολικό διαφορικό

Εκφράζει την μεταβολή της εξαρτημένης μεταβλητής (π.χ) z, όταν οι ανεξάρτητες μεταβλητές μεταβάλλονται κατά dx και dy αντίστοιχα.

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Άσκηση

Να βρεθεί το ολικό διαφορικό dz

$$z = x^4 + xy^3 + x^2y^4 = f(x,y)$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = 4x^3 + y^3 + 2xy^4$$

$$\frac{\partial f}{\partial y} = 3xy^2 + 4x^2y^3$$

$$dz = (4x^3 + y^3 + 2xy^4)dx + (3xy^2 + 4x^2y^3)dy$$

Άσκηση.

Να βρεθεί το ολικό διαφορικό

$$z = f(x,y) = x\ln y - y\ln x$$

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\ln y) - \frac{\partial}{\partial x}(y\ln x) =$$

$$= \ln y - \frac{y}{x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\ln y) - \frac{\partial}{\partial y}(y\ln x) =$$

$$= \frac{x}{y} - \ln x$$

$= \frac{x}{y} - \ln x$

$$dz = \left(\ln y - \frac{y}{x}\right)dx + \left(\frac{x}{y} - \ln x\right)dy$$

Ακρότατα συναρτήσεων 2 μεταβλητών

$f(x,y)$ λέμε ότι έχει στο σημείο (x_0, y_0) τοπικό μέγιστο αν $f(x,y) \leq f(x_0, y_0)$ για όλα τα σημεία (x,y) που ανήκουν σε μια περιοχή γύρω από το (x_0, y_0).

$f(x,y)$ έχει στο σημείο (x_0, y_0) τοπικό ελάχιστο αν $f(x,y) \geq f(x_0, y_0)$ για όλα τα σημεία (x,y) που ανήκουν σε μια περιοχή γύρω από το (x_0, y_0).

Αναζητώ τα κρίσιμα σημεία, δηλαδή $\Rightarrow$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial x}(x_0, y_0) = 0 \qquad \frac{\partial f}{\partial x}(x_0, y_0) = 0.$$

Θεώρημα.

Έστω ότι η συνάρτηση $f(x,y)$ έχει παραγώγους τουλάχιστον μέχρι 2ης τάξης και ότι (x_0, y_0) είναι ένα κρίσιμο σημείο, δηλαδή

$$\frac{\partial f(x_0, y_0)}{\partial x} = 0 \qquad \frac{\partial f(x_0, y_0)}{\partial y} = 0$$

Τότε αν θέσουμε

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2} = A, \quad \frac{\partial^2 f(x_0, y_0)}{\partial y^2} = B, \quad \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y} = C$$

Θα ισχύει ένα από τα παρακάτω.

I/ Αν $A \cdot B - C^2 > 0$, $A > 0 \Leftrightarrow f(x,y)$ έχει τοπικό ελάχιστο στο (x_0, y_0)

Αν $A \cdot B - C^2 > 0$, $A < 0 \Leftrightarrow f(x,y)$ έχει τοπικό μέγιστο

στο (x_0, y_0)

2/ Αν $A \cdot B - C^2 > 0$, $A < 0 \Leftrightarrow f(x,y)$ έχει τοπικό μέγιστο στο (x_0, y_0)

3/ Αν $A \cdot B - C^2 < 0 \Leftrightarrow f(x,y)$ δεν έχει τοπικό ακρότατο στο (x_0, y_0)

4/ Αν $A \cdot B - C^2 = 0 \Leftrightarrow f(x,y)$ άλλες φορές έχει και άλλες δεν έχει τοπικό ακρότατο στο (x_0, y_0).

Άσκηση

Να βρείτε τα ακρότατα της συνάρτησης

$$f(x,y) = x^2 + xy + y^2 - 3x - 6y$$

$$\frac{\partial f}{\partial x} = 2x + y - 3 \qquad \frac{\partial f}{\partial y} = x + 2y - 6$$

$$\left.\begin{aligned} \frac{\partial f(x_0,y_0)}{\partial x} = 0 &\Leftrightarrow 2x_0 + y_0 - 3 = 0 \\ \frac{\partial f(x_0,y_0)}{\partial y} = 0 &\Leftrightarrow x_0 + 2y_0 - 6 = 0 \end{aligned}\right\} \left\{\begin{aligned} 2x_0 + y_0 - 3 = 0 \\ -2x_0 - 4y_0 + 12 = 0 \end{aligned}\right\}$$

$$-3y_0 + 9 = 0 \Leftrightarrow y_0 = 3$$

$$x_0 = 6 - 2y_0 = 6 - 6 = 0$$

$(x_0, y_0) = (0, 3)$ κρίσιμο σημείο

$$\frac{\partial^2 f}{\partial x} = 2 \qquad \frac{\partial^2 f(x_0,y_0)}{\partial x} = A = 2 > 0$$

$$\frac{\partial^2 f}{\partial y} = 2 \qquad \frac{\partial^2 f(x_0,y_0)}{\partial y} = B = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 1 \qquad \frac{\partial^2 f(x_0,y_0)}{\partial x \partial y} = C = 1$$

$$A \cdot B - C^2 = 2 \cdot 2 - 1^2 = 4 - 1 = 3 > 0$$

$$A \cdot B - C^2 = 2 \cdot 2 - 1^2 = 4 - 1 = 3 > 0$$

Ξέρω ότι $A > 0$, άρα η $f(x,y)$ στο $(x_0, y_0) = (0,3)$ έχει τοπικό ελάχιστο.

Άσκηση

Να βρείτε τα ακρότατα της

$$z = f(x,y) = 2x - x^2 - y^2 - 4y - 4$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x - x^2 - y^2 - 4y - 4) =$$
$$= \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial x}(x^2) = \underline{2 - 2x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x - x^2 - y^2 - 4y - 4) =$$
$$= \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial y}(-4y) = \underline{-2y - 4}$$

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \Leftrightarrow 2 - 2x_0 = 0 \Leftrightarrow 2x_0 = 2 \Leftrightarrow x_0 = 1$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = 0 \Leftrightarrow -2y_0 - 4 = 0 \Leftrightarrow 2y_0 = -4 \Leftrightarrow y_0 = -2$$

Κρίσιμο σημείο είναι το $(x_0, y_0) = (1, -2)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2 - 2x) = -2.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-2y - 4) = -2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-2y - 4) = 0$$

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = A = -2 < 0$$

$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = B = -2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = C = 0$$

$$A \cdot B - C^2 = (-2)(-2) - 0^2 = 4 > 0$$

$A < 0$ άρα η $f(x,y)$ στο $(x_0, y_0) = (1,-2)$

έχει τοπικό μέγιστο.

Άσκηση

Δίνεται η συνάρτηση 2 μεταβλητών

$f(x,y) = ax^2 + by^2 - 8x + 6y + 5 \quad a, b, \in \mathbb{R}$

a) Να προσδιοριστούν οι σταθερές a, b : το σημείο $A(2,-1)$ να είναι πιθανό σημείο ακροτάτου (κρίσιμο σημείο) για την συνάρτηση $f(x,y)$

b) Στο σημείο $A(2,-1)$ η $f(x,y)$ παρουσιάζει τοπικό ακρότατο; Αν ναι τι είδους ακρότατο;

a) $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(ax^2 + by^2 - 8x + 6y + 5\right) = \frac{\partial}{\partial x}(ax^2) - \frac{\partial}{\partial x}(8x) =$

$= 2ax - 8$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(ax^2 + by^2 - 8x + 6y + 5\right) = \frac{\partial}{\partial y}(by^2) + \frac{\partial}{\partial y}(6y) =$

$= 2by + 6$

$\frac{\partial f}{\partial x}(x_0, y_0) = 0 \iff \frac{\partial f}{\partial x}(2,-1) = 0 \iff 2ax_0 - 8 = 0 \iff$

$\iff 2a2 - 8 = 0 \iff 4a - 8 = 0 \iff a = 8/4 = 2$

$\frac{\partial f}{\partial y}(x_0, y_0) = 0 \iff \frac{\partial f}{\partial y}(2,-1) = 0 \iff 2by_0 + 6 = 0 \iff$

$\iff 2b \cdot (-1) + 6 = 0 \iff 2b = 6 \iff b = 6/2 = 3$

Άρα $f(x,y) = 2x^2 + 3y^2 - 8x + 6y + 5$

$$\frac{\partial f}{\partial x} = 4x - 8$$

$$\frac{\partial f}{\partial y} = 6y + 6$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(4x-8) = 4$$

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0) = \frac{\partial^2 f}{\partial x^2}(2,-1) = A \iff \frac{\partial^2 f}{\partial x^2}(2,-1) = 4 = A > 0$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(6y+6) = 6$$

$$\frac{\partial^2 f}{\partial y^2}(x_0,y_0) = \frac{\partial^2 f}{\partial y^2}(2,-1) = B \iff \frac{\partial^2 f}{\partial y^2}(2,-1) = 6 = B$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(6y+6) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial x \partial y}(2,-1) = C = 0$$

$$\left.\begin{array}{l} A \cdot B - C^2 = 4 \cdot 6 - 0^2 = 24 > 0 \\ \text{όμως } A > 0 \end{array}\right\} \begin{array}{l} f(x,y) \text{ έχει τοπικό} \\ \text{ελάχιστο στο} \\ (x_0,y_0) = (2,-1) \end{array}$$

Συναρτησιακές ορίζουσες (Ιακωβιανές)

Θεωρούμε $f(x,y)$ κ' $g(x,y)$ παραγ. σε ανοικτό σύνολο $A \subseteq \mathbb{R}^2$

Ιακωβιανή f κ' g ως προς τις μεταβλητές x, y

Είναι η ορίζουσα

Ιακωβιανή ...
Είναι η ορίζουσα

$$D = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \frac{D(f,g)}{D(x,y)} \text{ ή } \frac{\partial(f,g)}{\partial(x,y)}$$

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

Ιδιότητες:

1/ $$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} \cdot \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(t_1, t_2, \ldots, t_n)} = \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(t_1, t_2, \ldots, t_n)}$$

2/ $$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} \cdot \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(f_1, f_2, \ldots, f_n)} = 1$$

3/ $$\frac{\partial(f,g)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(f,g)} = 1$$

4/ Αν $x = f(t,u)$ $y = g(t,u)$

$$dx \cdot dy = \left| \frac{D(x,y)}{D(t,u)} \right| dt\, du.$$

Παράδειγμα:

Αν $x = \rho\cos\theta$ $y = \rho\sin\theta$ (πολικές)

$$\frac{\partial x}{\partial \rho} = \frac{\partial}{\partial \rho}(x = \rho\cos\theta)$$

$$dx\, dy = \rho \cdot d\rho\, d\theta.$$

$$\frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} =$$

$D(\rho,\theta) \quad \left| \frac{\partial y}{\partial \rho} \quad \frac{\partial y}{\partial \theta} \right|$

$= \rho\cos\theta\cdot\cos\theta - (\sin\theta\cdot(-\rho\sin\theta)) =$

$= \rho\cdot\cos^2\theta + \rho\sin^2\theta = \rho(\cos^2\theta + \sin^2\theta) = \rho$

Άσκηση

$u=u(x,y)$, $v=v(x,y)$ $\quad w=w(x,y)$ N.δ.ο

$$\frac{D(u\cdot v^{-1},w)}{D(x,y)} = \frac{1}{v}\cdot\frac{D(u,w)}{D(x,y)} - \frac{u}{v^2}\frac{D(v,w)}{D(x,y)} \qquad \frac{\partial}{\partial x}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{\partial v}{\partial x}$$

$$J = \frac{D(uv^{-1},w)}{D(x,y)} = \begin{vmatrix} \frac{\partial(uv^{-1})}{\partial x} & \frac{\partial(uv^{-1})}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} \qquad \left(\begin{matrix} v^{-1} = \frac{1}{v} \\ u\cdot v^{-1} = \frac{u}{v} \end{matrix} \right)$$

$$\frac{\partial(uv^{-1})}{\partial x} = \frac{\partial\left(\frac{u}{v}\right)}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{1}{v} + u\cdot\frac{\partial\left(\frac{1}{v}\right)}{\partial x} = \frac{1}{v}\frac{\partial u}{\partial x} - \frac{u}{v^2}\frac{\partial v}{\partial x}$$

$$\frac{\partial(uv^{-1})}{\partial y} = \frac{\partial\left(\frac{u}{v}\right)}{\partial y} = \frac{\partial u}{\partial y}\cdot \;\ldots \qquad = \frac{1}{v}\frac{\partial u}{\partial y} - \frac{u}{v^2}\frac{\partial v}{\partial y}$$

$$J = \begin{vmatrix} \frac{1}{v}\frac{\partial u}{\partial x} - \frac{u}{v^2}\frac{\partial v}{\partial x} & \frac{1}{v}\frac{\partial u}{\partial y} - \frac{u}{v^2}\frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} =$$

$$= \left(\frac{1}{v}\frac{\partial u}{\partial x} - \frac{u}{v^2}\frac{\partial v}{\partial x}\right)\cdot\frac{\partial w}{\partial y} - \left(\frac{1}{v}\frac{\partial u}{\partial y} - \frac{u}{v^2}\frac{\partial v}{\partial y}\right)\cdot\frac{\partial w}{\partial x} =$$

$$= \frac{1}{v}\frac{\partial u}{\partial x}\cdot\frac{\partial w}{\partial y} - \frac{u}{v^2}\frac{\partial v}{\partial x}\cdot\frac{\partial w}{\partial y} - \frac{1}{v}\frac{\partial u}{\partial y}\cdot\frac{\partial w}{\partial x} + \frac{u}{v^2}\frac{\partial v}{\partial y}\cdot\frac{\partial w}{\partial x} =$$

$$= \frac{1}{v}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial w}{\partial x}\right) - \frac{u}{v^2}\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}\right) =$$

$$= \frac{1}{V}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial w}{\partial x}\right) - \frac{u}{V^2}\left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial w}{\partial x}\right)$$

$$= \frac{1}{v}\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} - \frac{u}{V^2}\begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix}$$

Άσκηση $\underset{u(x,y)}{u = x^2 + y^2}$, $\underset{v(x,y)}{v = x^3 + y^3}$

Ν.ο.ο $\frac{D(x,y)}{D(u,v)} = -\frac{1}{6xy(x-y)}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = 2x$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 2y$$
$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x^3+y^3) = 3x^2$$
$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x^3+y^3) = 3y^2$$

$$\frac{D(x,y)}{D(u,v)} \cdot \frac{D(u,v)}{D(x,y)} = 1$$

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 3x^2 & 3y^2 \end{vmatrix} =$$

$$= 2x \cdot 3y^2 - 2y \cdot 3x^2 = 6xy^2 - 6yx^2 =$$
$$= -6xy(x-y)$$

$$\frac{D(x,y)}{D(u,v)} = \frac{1}{\frac{D(u,v)}{D(x,y)}} = -\frac{1}{6xy(x-y)}$$

Παράγωγος κατά κατεύθυνση

$f(x,y)$ ορισμένη σε ανοιχτό σύνολο $A \subseteq \mathbb{R}^2$ και ένα μοναδιαίο διάνυσμα $\bar{v} = (v_1, v_2) = v_1 i + v_2 j$ του $\mathbb{R}^2$

Παράγωγος της f στο $P(x,y)$ κατά τη διεύθυνση του $\bar{v}$ ονομάζεται το όριο

$$\lim_{t \to 0} \frac{f(x + t v_1, y + t v_2) - f(x,y)}{t} = \frac{d f(P)}{d \bar{v}}$$

$$\lim_{t\to 0} \underbrace{\frac{f(x+tv_1, y+tv_2) - f(x,y)}{t}}_{} = \frac{df}{d\bar{v}}$$

συνάρτηση μεταβολής της f πάνω στην ευθεία
που ενώνει τα σημεία $P(x,y)$ κ' $P(x+tv_1, y+tv_2)$

$$\frac{df}{d\bar{v}} = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y} \qquad \bar{v} = (v_1, v_2) \quad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

∇f ανάδελτα της f ή gradf ή κλίση της f

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$\frac{df}{d\bar{v}} = \bar{v} \cdot \nabla f$$

$$\frac{d^2 f}{d\bar{v}^2} = \left(v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}\right)^{(2)} = v_1^2 \frac{\partial^2 f}{\partial x^2} + 2 v_1 v_2 \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + v_2^2 \frac{\partial^2 f}{\partial y^2}$$

Παράδειγμα

Να υπολογίσουμε κ' την 2η παράγωγο της $f(x,y) = x^3 + x^2 y$
κατά την διεύθυνση $\bar{v}(1,1)$ στο $P(2,3)$

μοναδιαίο διάνυσμα $\bar{v}_0 = \frac{\bar{v}}{|\bar{v}|} = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$|\bar{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\frac{\partial f}{\partial \bar{v}} = v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y} = \frac{\sqrt{2}}{2}(3x^2 + 2xy) + \frac{\sqrt{2}}{2} \cdot x^2$$

$$\frac{d^2 f}{d\bar{v}^2} = \left(v_1 \frac{\partial f}{\partial x} + v_2 \frac{\partial f}{\partial y}\right)^2 = v_1^2 \frac{\partial^2 f}{\partial x^2} + 2 v_1 v_2 \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + v_2^2 \frac{\partial^2 f}{\partial y^2} =$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 \cdot \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + 2 \cdot \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2} \cdot \frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} + \left(\frac{\sqrt{2}}{2}\right)^2 \cdot \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) =$$

$$= \frac{2}{4} \cdot \frac{\partial}{\partial x}(3x^2 + 2xy) + 2\,\frac{2}{4}(3x^2 + 2xy) \cdot x^2 + \frac{2}{4}\frac{\partial}{\partial y}(x^2) =$$

$$= \frac{1}{2}(6x + 2y) + (3x^2 + 2xy) \cdot x^2 + \frac{1}{2} \cdot 0 =$$

$= \frac{1}{2}(6x+2y) + (2x^2+2xy)\cdot x^2 + \frac{1}{2}\cdot 0 =$

$= \frac{1}{2}(6\cdot 2+3\cdot 2) + (3\cdot 2^2+2\cdot 2\cdot 3)\cdot 2^2 = \ldots$

$\ldots$

Ομογενείς συναρτήσεις

Μια συνάρτηση $f(x,y)$, ορισμένη σε ένα ανοιχτό σύνολο $A \subseteq \mathbb{R}^2$, είναι ομογενής βαθμού $m \in \mathbb{R}$ όταν $\forall x,y \in A$ $t>0$ ισχύει

$$f(tx,ty) = t^m f(x,y)$$

Παράδειγμα

$f(x,y) = 3x+5y$

$$f(tx,ty) = 3tx + 5ty = t(3x+5y) = t^1 f(x,y) \qquad m=1$$

Όταν είναι ομογενής $f(x,y)$ βαθμού m

a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = m f(x,y)$ (τύπος Euler)

b) $\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)^n = m(m-1)(m-2)\cdots(m-(n-1))\cdot f(x,y)$

Ανάπτυγμα Taylor-MacLaurin

Είναι η προσέγγιση μιας συνάρτησης με πολυώνυμα.

Αν μια $f(x,y)$ έχει συνεχείς μερικές παραγώγους μέχρι $n+1$

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\left[(x-x_0)\frac{\partial f(x_0,y_0)}{\partial x} + (y-y_0)\frac{\partial f(x_0,y_0)}{\partial y}\right] +$$

$$+ \frac{1}{2!}\left[(x-x_0)\frac{\partial f(x_0,y_0)}{\partial x} + (y-y_0)\frac{\partial f(x_0,y_0)}{\partial y}\right]^{(2)} + \cdots +$$

$$\ldots \frac{\partial f(x_0,y_0)}{}\Big]^{(n)} + R_n$$

$$+\frac{1}{4!}\left[(x-x_0)\frac{\partial f(x_0,y_0)}{\partial x}+(y-y_0)\frac{\partial f(x_0,y_0)}{\partial y}\right]^{(4)}+R_4$$

Πεπλεγμένες συναρτήσεις

Αν μια συν. $f(x,y,z)$ ορισμένη σε ένα ανοικτό σύνολο $A\subseteq \mathbb{R}^3$ ικανοποιεί

i) Η f κ' οι μερικές παράγωγοι αυτής είναι συνεχείς σε σε μια περιοχή του σημείου (x_0,y_0,z_0)

ii) $f(x_0,y_0,z_0)=0$ κ' $\frac{\partial f(x_0,y_0,z_0)}{\partial z}\neq 0$

τότε η $f(x,y,z)=0$ έχει μοναδική λύση την $z=g(x,y)$ σε περιοχή του (x_0,y_0) με $z_0=g(x_0,y_0)$.

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}=-\frac{f_x}{f_z} \quad κ' \quad \frac{\partial z}{\partial y}=-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}=-\frac{f_y}{f_z}$$

$$\boxed{f_x+f_z\cdot z_x=0} \qquad \boxed{f_y+f_z\cdot z_y=0}$$

$$z_x=-\frac{f_x}{f_z}\Leftrightarrow$$

$$f_z z_x=-f_x\Leftrightarrow$$

$$f_x+f_z z_x=0$$

$$\boxed{\begin{aligned}&f_x+f_z z_x=0\\&f_y+f_z z_y=0\end{aligned}}$$

$$f_{xx}+2f_{xz}z_x+f_{zz}z_x^2+f_z z_{xx}=0$$

$$f_{yy}+2f_{yz}z_y+f_{zz}\cdot z_y^2+f_z z_{yy}=0$$

$f_{yy} + 2f_{yz}z_y + f_{zz}\cdot z_y + f_z z_{yy}$ ...

f_x επειδή είναι πάλι συνάρτηση $f_{xx} + f_{xz}\cdot z_z$

$$\frac{\partial}{\partial x}(f_z)\cdot z_x + f_z\cdot\frac{\partial}{\partial x}(z_x) = (f_{xx} + f_{zz}z_x)z_x + f_z z_{xx}$$

Διανυσματική συνάρτηση

t παίρνει τιμές στο $I=[a,b]$ $t\in I$ κ' $\forall t\in I$ αντιστοιχίζουμε

ένα διάνυσμα $\bar{r}(t)$

$$\bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}$$

$$\lim_{t\to t_0}\bar{r}(t) = \underbrace{\left(\lim_{t\to t_0}x(t)\right)}_{a_1}\cdot\bar{i} + \underbrace{\left(\lim_{t\to t_0}y(t)\right)}_{a_2}\bar{j} + \underbrace{\left(\lim_{t\to t_0}z(t)\right)}_{a_3}\cdot\bar{k}$$

$$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$

Η $\bar{r}(t)$ είναι μια καμπύλη με παραμετρικές εξισώσεις $x(t), y(t), z(t)$

- κλειστή $t\in[a,b]$ $\bar{r}(a) = \bar{r}(b)$
- απλή $t_1, t_2\in[a,b]$ $\bar{r}(t_1)\neq\bar{r}(t_2)$

Παραγώγιση

$t\in I$ $\lim_{t\to t_0}\dfrac{\bar{r}(t) - \bar{r}(t_0)}{t - t_0}$

$$\frac{d\,\bar{r}(t_0)}{dt} = \dot{\bar{r}}(t) = x'(t)\,\bar{i} + y'(t)\,\bar{j} + z'(t)\,\bar{k}$$

$\bar{r}(t)$ κ' $t = t(s)$

$$\frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt}\cdot\frac{dt}{ds}$$

Κανόνες παραγώγισης

Αν έχω διαν. συν. $\bar{r}_1(t)$, $\bar{r}_2(t)$

1/ $\frac{d}{dt}(\bar{r}_1+\bar{r}_2) = (\bar{r}_1+\bar{r}_2)^{\cdot} = \dot{\bar{r}}_1 + \dot{\bar{r}}_2$

2/ $\frac{d}{dt}(\bar{r}_1\cdot\bar{r}_2) = (\bar{r}_1\cdot\bar{r}_2)^{\cdot} = \dot{\bar{r}}_1\bar{r}_2 + \bar{r}_1\cdot\dot{\bar{r}}_2$ (εσωτερικό γινόμενο)

Εξωτερικό γινόμενο (διανυσματικό)

$\bar{i}$, $\bar{j}$, $\bar{k}$ βάση

$i = j\times k \Rightarrow k\times j = -i$

$j = k\times i \Rightarrow i\times k = -j$

$k = i\times j \Rightarrow j\times i = -k$

Αν έχω $\bar{u} = (u_1, u_2, u_3)$

$\bar{v} = (v_1, v_2, v_3)$

$$\bar{u}\times\bar{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i\cdot(-1)^{1+1}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + j(-1)^{1+2}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k\cdot(-1)^{1+3}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

3/ $\frac{d}{dt}(\bar{r}_1\times\bar{r}_2) = (\bar{r}_1\times\bar{r}_2)^{\cdot} = \dot{\bar{r}}_1\times\bar{r}_2 + \bar{r}_1\times\dot{\bar{r}}_2$

4/ $\frac{d}{dt}(\bar{r}_1\bar{r}_2\bar{r}_3) = (r_1r_2r_3)^{\cdot} = \dot{\bar{r}}_1\bar{r}_2\bar{r}_3 + \bar{r}_1\dot{\bar{r}}_2\bar{r}_3 + \bar{r}_1\bar{r}_2\dot{\bar{r}}_3$

5/ $\frac{d}{dt}(\bar{r}_1\times(\bar{r}_2\times\bar{r}_3)) = (\bar{r}_1\times(\bar{r}_2\times\bar{r}_3))^{\cdot} =$

$= \dot{\bar{r}}_1\times(\bar{r}_2\bar{r}_3) + \bar{r}_1\times(\dot{\bar{r}}_2\times\bar{r}_3) + \bar{r}_1\times(\bar{r}_2\times\dot{\bar{r}}_3)$

6/ $\frac{d}{dt}(f\cdot\bar r) = (f\bar r)' = \frac{df}{dt}\bar r + f\cdot\dot{\bar r}$ $\quad (f = f(t))$

7/ $\frac{d}{dt}(c\bar r) = (c\bar r)' = c\cdot\dot{\bar r}$ $\quad$ (c σταθερά)

Δεύτερη παράγωγος

$$\frac{d^2\bar r}{dt^2} = \ddot{\bar r}(t) = \frac{d}{dt}\left(\dot{\bar r}(t)\right) \Rightarrow \frac{d^\nu r}{dt^\nu} = \bar r^{(\nu)}$$

$|\bar r| =$ σταθερό $\iff \bar r \perp \dot{\bar r} \iff \bar r\cdot\dot{\bar r} = 0$

- Το $\bar r$ έχει σταθερή διεύθυνση $\iff \bar r // \dot{\bar r}$

$\bar r \times \dot{\bar r} = 0$

- Ταχύτητα $\quad \bar U(t) = \dot{\bar r}(t)$

Επιτάχυνση $\quad \bar a(t) = \ddot{\bar r}(t) = \dot{\bar U}(t)$

Ολοκληρώματα διανυσματικής συνάρτησης

$\bar r(t) = x(t)\bar i + y(t)\bar j + z(t)\bar k$

a) $\int \bar r(t)\,dt = \left(\int x(t)dt\right)\bar i + \left(\int y(t)\,dt\right)\bar j + \left(\int z(t)dt\right)\bar k$

b) $\int_a^b r(t)\,dt = \left(\int_a^b x(t)\,dt\right)\bar i + \left(\int_a^b y(t)\,dt\right) j + \left(\int_a^b z(t)\,dt\right)\bar k$

Ισχύουν

1/ $\int \dot{\bar r}(t)\,dt = \bar r(t) + \bar c$ $\quad$ $\bar c$ σταθερό

2/ $\int_a^b \dot{\bar r}(t)\,dt = \bar r(b) - \bar r(a)$

3/ $\int^b \bar a\cdot\bar r(t)\,dt = \bar a\int^b \bar r(t)\,dt$

3/ $\int_a^b \bar{a}\cdot\bar{r}(t)\,dt = \bar{a}\int_a^b \bar{r}(t)\,dt$

4/ $\int_a^b \bar{a}\times\bar{r}(t)\,dt = \bar{a}\times\int_a^b \bar{r}(t)\,dt$

$\Big\}$ $\bar{a}$ σταθερό

Συνοδεύον τρίεδρο - εφαπτόμενο πεδίο - τύπος Frenet - καμπυλότητα και στρέψη καμπύλης.

Θεωρούμε μια καμπύλη $C: \bar{r}(t)$ $t\in[a,b]$

Αν η καμπύλη είναι ομαλή (δηλαδή $\dot{\bar{r}}(t)\neq\bar{0}$ $\forall t$ & $\bar{r}(t)$ συνεχής στο (a,b)) & $\exists$ $\lim_{t\to a}\dot{\bar{r}}(t)$ $\lim_{t\to b}\dot{\bar{r}}(t)$ τότε C λέγεται λεία.

($\dot{\bar{r}}(t_0)\neq\bar{0}$ δηλαδή οι $\dot{x}(t_0), \dot{y}(t_0), \dot{z}(t_0)$ δεν είναι όλες μηδέν τότε στο t_0 λέγεται ομαλό σημείο

Όταν όλα τα x, y, z $(\epsilon\cdot)\neq 0$ τότε η C λέγεται ομαλή)

$C: \bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}$ είναι λεία τότε το μήκος του τόξου της καμπύλης με άκρα στα $A(\bar{r}(a))$, $B(\bar{r}(b))$

$$s=\int_a^b |\dot{\bar{r}}(t)|\,dt = \int_a^b \sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2+(\dot{z}(t))^2}\,dt$$

από $t_0 \to t$

$$s=\int_{t_0}^t |\dot{\bar{r}}(t)|\,dt = \int_{t_0}^t \sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2+(\dot{z}(t))^2}\,dt$$

Αν $\bar{r}=\bar{r}(s)$ $\quad s\in[0,S]$

$$\bar{r}(s) = x(t(s))\,\bar{i} + y(t(s))\,\bar{j} + z(t(s))\,\bar{k}$$

Ισχύει

$$\frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt}\cdot\frac{dt}{ds} = \frac{\frac{dr}{dt}}{\frac{ds}{dt}} = \frac{\dot{\bar{r}}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \frac{\dot{\bar{r}}}{|\dot{\bar{r}}|}$$

$$\frac{d\bar{r}}{ds} = \frac{d\bar{r}}{dt}\cdot\frac{dt}{ds} = \frac{\frac{dr}{dt}}{\frac{ds}{dt}} = \frac{v}{\sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2}} = \frac{v}{|\dot{r}|}$$

$$\bar{\varepsilon}_0(s) = \frac{d\bar{r}}{ds} = \frac{\dot{\bar{r}}(t)}{|\dot{r}(t)|}$$ μοναδιαίο εφαπτόμενο διαν.

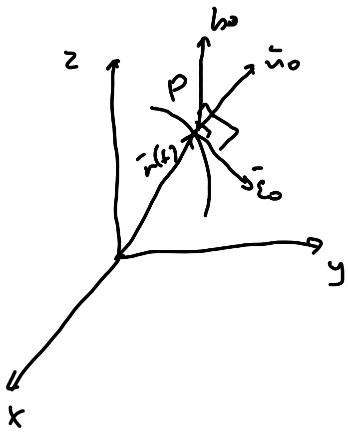
$$\frac{d\bar{\varepsilon}_0}{ds} \Rightarrow \bar{n}_0 = \frac{1}{k}\frac{d\bar{\varepsilon}_0}{ds}$$ πρώτη κάθετος

k = καμπυλότητα της C σε ένα P

$\frac{1}{k} = \rho$ ακτίνα καμπυλότητας

$|\bar{\varepsilon}_0| = 1 \qquad \varepsilon_0 \perp \frac{d\bar{\varepsilon}_0}{ds} \qquad \bar{\varepsilon}_0 \perp \bar{n}_0$

$\bar{b}_0 = \bar{\varepsilon}_0 \times \bar{n}_0$ δεύτερη κάθετος



$\bar{\varepsilon}_0, \bar{n}_0, \bar{b}_0$ ορίζουν ένα ορθοκανονικό δεξιόστροφο τρισορθογώνιο σύστημα συντ.

- Το επίπεδο που ορίζουν τα $\bar{\varepsilon}_0, \bar{n}_0$ ονομάζεται εγγύτατο επίπεδο της C στο σημείο P

$(\bar{R} - \bar{r})\,\bar{b}_0 = 0$ ή $\bar{R} = \bar{r} + \lambda\bar{\varepsilon}_0 + \mu\bar{n}_0 \qquad \lambda, \mu \in \mathbb{R}$

$\bar{R}$ = διάνυσμα θέσης ενός τυχαίου σημείου στο εγγύτατο

- Το επίπεδο που ορίζουν τα $\bar{n}_0, \bar{b}_0$ ονομάζεται κάθετο επίπεδο της C στο P

$(\bar{R} - \bar{r})\,\bar{\varepsilon}_0 = 0$ ή $\bar{R} = \bar{r} + \lambda\bar{n}_0 + \mu\bar{b}_0 \qquad \lambda, \mu \in \mathbb{R}$.

- Το επίπεδο που ορίζουν τα $\bar{\varepsilon}_0, \bar{b}_0$ ονομάζεται ευθειοποιούν επίπεδο της C στο P

$(\bar{R}-\bar{r})\bar{n}_0=0$ ή $\bar{R}=\bar{r}+\lambda\bar{\varepsilon}_0+\mu\bar{b}_0$ $\lambda,\mu\in\mathbb{R}$

- Για μια λεία καμπύλη $\bar{r}=\bar{r}(s)$ $s\in[0,S]$

$$\frac{d\bar{\varepsilon}_0}{ds}=k\bar{n}_0\ ,\ \frac{d\bar{b}_0}{ds}=-\sigma\bar{n}_0\ ,\ \frac{d\bar{n}_0}{ds}=\sigma\bar{b}_0-k\bar{\varepsilon}_0 \quad \text{τύποι Frenet}$$

σ = στρέψη της C

$r=\frac{1}{\sigma}$ ακτίνα στρέψης.

- Υπολογισμός της καμπυλότητας

$$k=\frac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}=\frac{\sqrt{\begin{vmatrix}\dot{x}&\dot{y}\\\ddot{x}&\ddot{y}\end{vmatrix}^2+\begin{vmatrix}\dot{y}&\dot{z}\\\ddot{y}&\ddot{z}\end{vmatrix}^2+\begin{vmatrix}\dot{z}&\dot{x}\\\ddot{z}&\ddot{x}\end{vmatrix}^2}}{\sqrt{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^3}}$$

$\bar{r}(t)=(x(t),y(t),z(t))$

Αν η καμπύλη είναι επίπεδη

$$k=\frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{\sqrt{(\dot{x}^2+\dot{y}^2)^3}}$$

- Υπολογισμός στρέψης

$$\sigma=\frac{|\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}|}{|\dot{\bar{r}}\times\ddot{\bar{r}}|^2}=\frac{\begin{vmatrix}\dot{x}&\dot{y}&\dot{z}\\\ddot{x}&\ddot{y}&\ddot{z}\\\dddot{x}&\dddot{y}&\dddot{z}\end{vmatrix}}{\begin{vmatrix}\dot{x}&\dot{y}\\\ddot{x}&\ddot{y}\end{vmatrix}^2+\begin{vmatrix}\dot{y}&\dot{z}\\\ddot{y}&\ddot{z}\end{vmatrix}^2+\begin{vmatrix}\dot{z}&\dot{x}\\\ddot{z}&\ddot{x}\end{vmatrix}^2}$$

Μια καμπύλη είναι επίπεδη $\Leftrightarrow$ $\sigma=0$ $|\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}|=0$

- Υπολογισμός του $\bar{b}_0$

$$b_0=\frac{\dot{\bar{r}}\times\ddot{\bar{r}}}{|\dot{\bar{r}}\times\ddot{\bar{r}}|}$$

<u>Διαφορικοί τελεστές grad, div, rot</u>

Διαφορικοί τελεστές grad, div, rot

Έστω $f(x,y,z)$ βαθμωτή συνάρτηση

Κλίση της f ονομάζεται το διάνυσμα

$$\nabla f = \operatorname{grad} f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k} \quad (\nabla \text{ ανάδελτα})$$

Αν $f = f(x,y)$

$$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}$$

για βαθμωτές συναρτήσεις

$$\nabla(f+g) = \nabla f + \nabla g$$

$$\nabla(\lambda f + \mu g) = \lambda \nabla f + \mu \nabla g$$

$$\nabla(f \cdot g) = g\nabla f + f\nabla g$$

για μια συνάρτηση $f = g(u)$ $\quad u = u(x,y,z)$

$$\nabla f = \nabla g(u) = \frac{\partial g(u)}{\partial x}\bar{i} + \frac{\partial g(u)}{\partial y}\bar{j} + \frac{\partial g(u)}{\partial z}\bar{k}$$

$$= \frac{\partial g}{\partial u}\frac{\partial u}{\partial x}\bar{i} + \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y}\bar{j} + \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial z}\cdot\bar{k} = g'(u)\nabla u$$

$$\nabla = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$$

$$\bar{v} = (v_1, v_2, v_3)$$

$$\frac{\partial f}{\partial v} = v_1\frac{\partial f}{\partial x} + v_2\frac{\partial f}{\partial y} + v_3\frac{\partial f}{\partial z} = \bar{v}\cdot\nabla f.$$

- Θεωρούμε τη διανυσματική συνάρτηση

$$\bar{F} = \bar{F}(x,y,z) = F_1(x,y,z)\bar{i} + F_2(x,y,z)\bar{j} + F_3(x,y,z)\bar{k}$$

Απόκλιση της $\bar{F}$ στο $P(x,y,z)$

$$\operatorname{div}\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

είναι το εσωτερικό γινόμενο του

$$\bar{\nabla} = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k} \quad \text{με το} \quad \bar{F}$$

$$\operatorname{div}\bar{F} = \nabla \bar{F} = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right) \cdot \left(F_1\bar{i} + F_2\bar{j} + F_3\bar{k}\right)$$

Στροφή ή περιστροφή μιας $\bar{F}$

$$\bar{F} = \bar{F}(x,y,z) = F_1(x,y,z)\,\bar{i} + F_2(x,y,z)\,\bar{j} + F_3(x,y,z)\,\bar{k}$$

είναι το διάνυσμα

$$\operatorname{rot}\bar{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\bar{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\bar{j} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\bar{k}$$

ή

$$\operatorname{rot}\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

//

Αν η $\bar{F}(x,y) = F_1(x,y)\,\bar{i} + F_2(x,y)\,\bar{j}$

τότε $\operatorname{rot}\bar{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ //

Αν έχουμε μια επιφάνεια $S: f(x,y,z)=0$

τότε το διάνυσμα

$$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$ είναι κάθετο στο $P(x,y,z)$

Κάποιες ιδιότητες

1/ $\operatorname{div}(\operatorname{grad} f) = \nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ $\quad f(x,y,z)$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ λέγεται τελεστής του Laplace

$\nabla^2 f$ Λαπλασιανή της f $\qquad \Delta = \nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$

2/ $rot(grad f) = \nabla \times \nabla f = 0$, δηλαδή η στροφή της κλίσης είναι μηδέν.

3/ $div(rot \bar{F}) = \nabla \cdot (\nabla \times \bar{F}) = 0$, δηλαδή η απόκλιση της στροφής είναι μηδέν.

__Πεδία στοιχεία__

- Ένα υποσύνολο του τρισδιάστατου χώρου λέγεται βαθμωτό πεδίο (ή αριθμητικό πεδίο), όταν σε κάθε σημείο του αντιστοιχεί ένας πραγματικός αριθμός.

Παράδειγμα: Σε χώρο δωματίου για τη θερμοκρασία του δωματίου σε διάφορα σημεία

- Ένα υποσύνολο του R^3 λέγεται διανυσματικό πεδίο, όταν σε κάθε σημείο του αντιστοιχεί ένα διάνυσμα.

Παράδειγμα: Το πεδίο της ηλεκτρικής έντασης.

- Ένα διανυσματικό πεδίο $\bar{F} = \bar{F}(x,y,z)$ λέγεται σωληνοειδές όταν $div \bar{F} = 0$, ενώ λέγεται αστρόβιλο όταν ισχύει $rot \bar{F} = \bar{0}$

__Φυσική ερμηνεία της απόκλισης.__

Το διανυσματικό πεδίο των ταχυτήτων ενός ρευστού είναι

$$\bar{V}(x,y,z) = V_1(x,y,z)\bar{i} + V_2(x,y,z)\bar{j} + V_3(x,y,z)\bar{k}$$

$V_1(x,y,z) =$ η ταχύτητα του ρευστού κατά την κατεύθυνση του άξονα Ox.

$V_2(x,y,z) =$ —//— —//— —//— άξονα Oy.

$V_3(x,y,z) =$ —//— //— —//— άξονα Oz.

Αν σε κάποιο σημείο του ρευστού υπάρχει πηγή τότε

Αν σε κάποιο σημείο του ρευστού υπάρχει πηγή τότε η απόκλιση είναι θετική.
Αν σε κάποιο σημείο του ρευστού υπάρχει απώλεια τότε η απόκλιση είναι αρνητική.
Αν η ροή του ρευστού είναι σταθερή τότε η απόκλιση είναι μηδέν σε κάθε σημείο της μάζας του.

<u>Φυσική ερμηνεία της στροφής</u>

Αν πάρουμε για παράδειγμα το διανυσματικό πεδίο του ρευστού με την ταχύτητα, τότε η στροφή του πεδίου σε ένα σημείο P είναι ένα μέτρο της τοπικής περιστροφής της μάζας του ρευστού στο P.
Αν στο P υπάρχει στροβιλισμός τότε η στροφή δεν είναι μηδέν ενώ αν δεν υπάρχει τότε είναι μηδέν.

<u>Ύπαρξη δυναμικού με δοσμένη κλίση</u>

$\bar{r} = \bar{r}(x,y,z) = P(x,y,z)\,\bar{i} + Q(x,y,z)\,\bar{j} + R(x,y,z)\,\bar{k}$
και ζητείται να βρεθεί το βαθμωτό πεδίο $f(x,y,z)$ ώστε η κλίση να είναι $\bar{r}$

$$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k} = \bar{r}$$

Όταν υπάρχει η f τότε το διανυσματικό πεδίο $\bar{r}$ έχει ένα δυναμικό f (συνάρτηση δυναμικού f), αυτό ισχύει όταν:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Το πεδίο που προέρχεται από το δυναμικό ονομάζεται συντηρητικό σε κάθε σημείο μιας ορθογώνιας περιοχής.
Καθώς $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$

Καθώς $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$

θα ισχύει $\nabla f = Pdx + Qdy + Rdz$ λέγεται τέλειο διαφορικό

Η συνάρτηση $f(x,y,z)$ είναι

$f(x,y,z) = f_1(x,y,z) + f_2(y,z) + f_3(z)$ όπου

$f_1(x,y,z) = \int P dx$ (y, z σταθερά)

$f_2(y,z) = \int \left(Q - \frac{\partial f_1}{\partial y}\right) dy$ (z σταθερά)

$f_3(z) = \int \left(R - \frac{\partial f_1}{\partial z} - \frac{\partial f_2}{\partial z}\right) dz$

Παράδειγμα

Θα βρούμε τη δυναμική του διανυσματικού πεδίου

$\bar{r} = \frac{2x}{z}\bar{i} + 2yz\bar{j} + \left(y^2 - \frac{x^2}{z^2}\right)\bar{k}$ σε μια ορθογώνια

περιοχή όπου δεν περιέχει σημεία το xy επίπεδο $(y \cdot z \neq 0)$

$P = \frac{2x}{z}$, $Q = 2yz$, $R = y^2 - \frac{x^2}{z^2}$

$$\left.\begin{array}{l} \frac{\partial P}{\partial y} = 0 \\ \frac{\partial Q}{\partial x} = 0 \end{array}\right\} \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0$$

$$\left.\begin{array}{l} \frac{\partial P}{\partial z} = 2x\left(-\frac{1}{z^2}\right) = -\frac{2x}{z^2} \\ \frac{\partial R}{} = -\frac{2x}{} \end{array}\right\} \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = -\frac{2x}{z^2}$$

$$\frac{\partial R}{\partial x} = -\frac{2x}{z^2}$$

$$\left.\begin{aligned}\frac{\partial Q}{\partial z} &= 2y\\ \frac{\partial R}{\partial y} &= 2y\end{aligned}\right\} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 2y$$

$$f(x,y,z) = f_1(x,y,z) + f_2(y,z) + f_3(z)$$

$$f_1(x,y,z) = \int P\,dx = \int \frac{2x}{z}\,dx = \frac{2}{z}\int x\,dx + \frac{2}{z}\frac{x^2}{2} + c_1 = \frac{x^2}{z} + c_1$$

$$f_2(y,z) = \int\left(Q - \frac{\partial f_1}{\partial y}\right)dy = \int(2yz - 0)\,dy = y^2 z + c_2$$

$$f_3(z) = \int\left(R - \frac{\partial f_1}{\partial z} - \frac{\partial f}{\partial z}\right)dz = \int\left(\left(y^2 - \frac{x^2}{z^2}\right) - \left(-\frac{x^2}{z^2}\right) - y^2\right)dz + c_3 =$$

$$= \int\left(\cancel{y^2} - \cancel{\frac{x^2}{z^2}} + \cancel{\frac{x^2}{z^2}} - \cancel{y^2}\right)dz + c_3 = 0 + c_3$$

$$f(x,y,z) = \frac{x^2}{z} + y^2 z + c \qquad c = c_1 + c_2 + c_3$$

// Αν έχουμε το διανυσματικό πεδίο

$$\bar{F} = P(x,y)\,\bar{i} + Q(x,y)\,\bar{j}$$

σε ένα ανοικτό ορθογώνιο xy-επίπεδο, τότε το δυναμικό f υπάρχει $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$f(x,y,z) = f_1(x,y) + f_2(y)$$

$$f(x,y,z) = f_1(x,y) + f_2(y)$$

$$f_1(x,y) = \int P\,dx$$

$$f_2(y) = \int \left(Q - \frac{\partial f_1}{\partial y}\right) dy \quad //$$

Άσκηση

Αν $\bar{a}, \bar{b}$ δοσμένα σταθερά διανύσματα, να αποδείξουμε ότι το διανυσματικό πεδίο

$$\bar{r} = (\cos(kt))\bar{a} + (\sin(kt))\bar{b} \qquad k \neq 0 \text{ σταθ}$$

έχει σταθερή διεύθυνση $c=1$ $\quad \bar{a} // \bar{b}$

$\bar{r} // \dot{\bar{r}}$ δηλαδή $\bar{r} \times \dot{\bar{r}} = \bar{0}$

$$\dot{\bar{r}} = (\cos(kt))^{\cdot}\bar{a} + (\cos(kt))\cdot\dot{\bar{a}} + (\sin(kt))^{\cdot}\cdot\bar{b} + (\sin(kt))\cdot\dot{\bar{b}}$$

$$= -k\sin(kt)\cdot\bar{a} + \cos(kt)\cdot 0 + k\cos(kt)\cdot\bar{b} + \sin(kt)\cdot\bar{0}$$

$$= -k\sin(kt)\bar{a} + k\cos(kt)\bar{b}$$

$$\bar{r} \times \dot{\bar{r}} = (\cos(kt)\bar{a} + \sin(kt)\bar{b}) \times (-k\sin(kt)\bar{a} + k\cos(kt)\bar{b})$$

$$= (-k\sin(kt))\cdot\cos(kt)\,\bar{a}\times\bar{a} + k\cos^2(kt)\,\bar{a}\times\bar{b} -$$

$$- k\sin^2(kt)\,\bar{b}\times\bar{a} + k\sin(kt)\cdot\cos(kt)\,\bar{b}\times\bar{b}$$

$$\bar{a}\times\bar{a} = 0 = \bar{b}\times\bar{b} \qquad \bar{b}\times\bar{a} = -\bar{a}\times\bar{b}$$

$$= k(\cos^2(kt) + \sin^2(kt))\,\bar{a}\times\bar{b} = k(\bar{a}\times\bar{b})$$

$$\bar{a}\times\bar{b} = \bar{0} \text{ όταν } \bar{a}//\bar{b}$$

Άσκηση

Άσκηση

Να αποδ. ότι η καμπύλη

$c: \bar{r} = (2t^2+3t+1)\bar{i} + (5t^2-2t+2)\bar{j} + (1-t^2)\bar{k} \quad t\in\mathbb{R}$

είναι επίπεδη κ' η εξίσωση.

Η καμπύλη είναι επίπεδη όταν η στρέψη είναι 0

Σχέση $\left|\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}\right| = 0$

$\dot{\bar{r}} = (4t+3,\ 10t-2,\ -2t)$

$\ddot{\bar{r}} = (4,\ 10,\ -2)$

$\dddot{\bar{r}} = (0,\ 0,\ 0)$

$$\left|\dot{\bar{r}}\cdot\ddot{\bar{r}}\cdot\dddot{\bar{r}}\right| = \begin{vmatrix} 4t+3 & 10t-2 & -2t \\ 4 & 10 & -2 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Άσκηση

Να βρείτε τους πραγματικούς αριθμούς a, b, c, ώστε το διανυσματικό πεδίο

$\bar{r} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$

να είναι αστρόβιλο.

$\operatorname{rot}\bar{r} = \bar{0}$

$$\operatorname{rot}\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} =$$

$$\left| (x+2y+az) \quad (bx-3y-z) \quad (4x+cy+2z) \right|$$

$$= \left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z) \right) \cdot \bar{i} \cdot (-1)^{1+1} +$$

$$+ \left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az) \right) \cdot \bar{j} \, (-1)^{1+2}$$

$$+ \left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az) \right) \cdot \bar{k} \cdot (-1)^{1+3} =$$

$$(c+1) \cdot \bar{i} - (4-a)\,\bar{j} + (b-2)\,\bar{k}$$

θv $rot \bar{r} = \bar{0}$

$c+1=0$
$4-a=0$
$b-2=0$