Μιγαδικοί Αριθμοί Εισαγωγή

Friday, October 9, 2020 4:26 PM

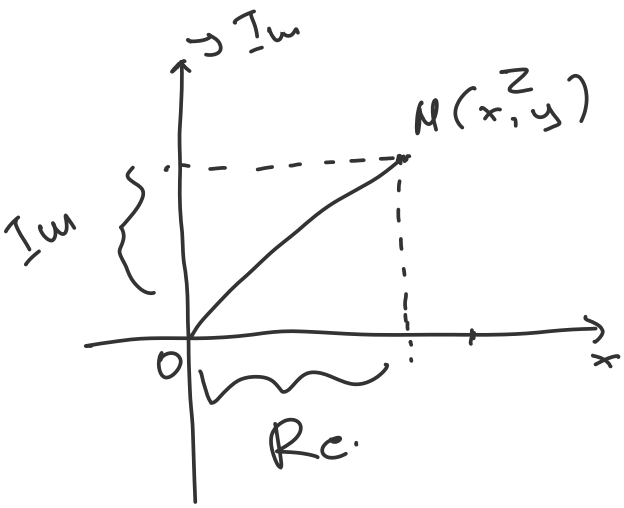
Το σύνολο C των μιγαδικών αριθμών είναι ένα υπερσύνολο του συνόλου R των πραγματικών αριθμών.
Οι πράξεις της πρόσθεσης και του πολλαπλασιασμού έχουν τους ίδιους κανόνες όπως στους πραγματικούς.
Υπάρχει ένα στοιχείο i τέτοιο ώστε $i^2 = -1$.
Κάθε στοιχείο z γράφεται κατά μοναδικό τρόπο με μορφή $z = a + bi$, με a, b $πραγματικοί$.

$z = a + bi \quad , a, b \in R$

$Re(z) = a \qquad Im(z) = b$

$Re(z) = 0 \Rightarrow$ Φανταστικό Αριθμό

$Im(z) = 0 \Rightarrow$ Πραγμ. Αριθ.



$z = x + iy$

$a = a + 0i$

$bi = 0 + bi$

Ισότητα $\quad z_1 = a_1 + b_1 i$ κ' $z_2 = a_2 + b_2 i$

$z_1 = z_2 \Leftrightarrow \left.\begin{array}{l} a_1 = a_2 \\ b_1 i = b_2 i \end{array}\right\} \begin{array}{l} a_1 = a_2 \\ b_1 = b_2 \end{array}$

Πρόσθεση $\quad z_1 = \alpha + \beta i$ κ' $z_2 = \gamma + \delta i$

$z_1 + z_2 = (\alpha + \beta i) + (\gamma + \delta i) = \alpha + \beta i + \gamma + \delta i =$

$= (\alpha + \gamma) + (\beta + \delta) i$

Ουδέτερο στοιχείο $\quad 0 = 0 + 0i$

Αντίθετος $\quad z = \alpha + \beta i \quad -z = -\alpha - \beta i$

Αφαίρεση

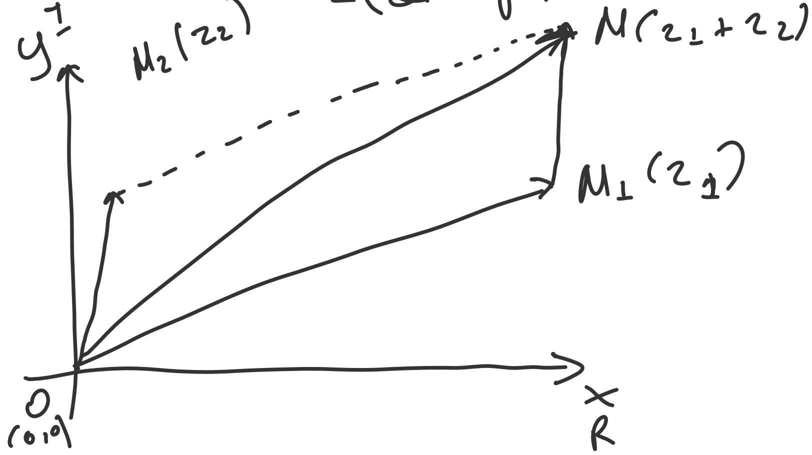
Αφαίρεση

$z_1 = \alpha + \beta i \qquad z_2 = \gamma + \delta i$

$z_1 - z_2 = (\alpha + \beta i) - (\gamma + \delta i) =$

$= \alpha + \beta i - \gamma - \delta i =$

$= (\alpha - \gamma) + (\beta - \delta) i$

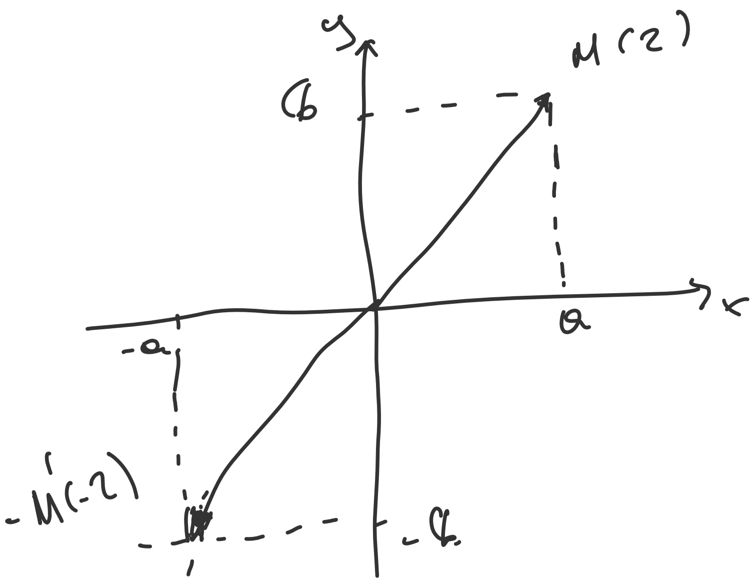


$z_1 = \alpha + \beta i$

$z_2 = \gamma + \delta i$

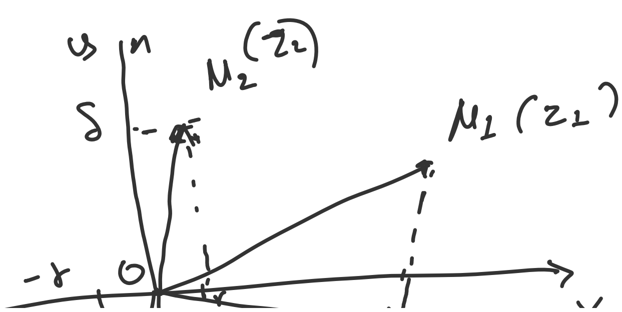
$z = \alpha + \beta i$

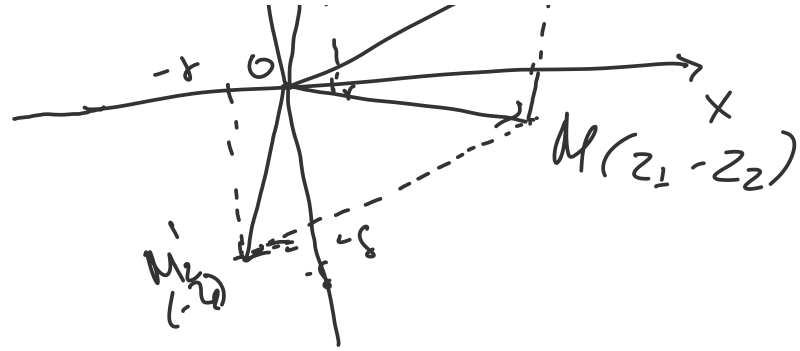
$-z = -\alpha - \beta i$



$z_1 = \alpha + \beta i \quad z_2 = \gamma + \delta i$

$z_1 - z_2 \quad -z_2 = -\gamma - \delta i$





Πολλαπλασιασμός

$z_1 = \alpha + \beta i \quad , z_2 = \gamma + \delta i \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R} \qquad i^2 = -1$

$z_1 \cdot z_2 = (\alpha + \beta i)(\gamma + \delta i) = \alpha\gamma + \alpha\delta i + \beta\gamma i + \beta\delta \cdot i^2 =$

$= \alpha\gamma + \alpha\delta i + \beta\gamma i - \beta\cdot\delta =$

$= \underset{\mathbb{R}}{(\alpha\gamma - \beta\delta)} + \underset{I}{(\alpha\delta + \beta\gamma)} i$

Ουδέτερο στοιχείο $\quad 1 + 0i$

Συμμετρικό ή αντίστροφο

$z \cdot \frac{1}{z} = 1 \qquad z = \alpha + \beta i \quad , \frac{1}{z} = x + yi$

$z \cdot \frac{1}{z} = 1 \Leftrightarrow (\alpha + \beta i)(x + yi) = 1 \Leftrightarrow$

$\alpha x + \alpha y i + \beta x i + \beta y i^2 = 1 \Leftrightarrow$

$\alpha x + \alpha y i + \beta x i - \beta y = 1 \Leftrightarrow$

$(\alpha x - \beta y) + (\alpha y + \beta x) i = 1 \Leftrightarrow$

$(\alpha x - \beta y) + (\alpha y + \beta x) i = 1 + 0i$

$\left.\begin{array}{l} \alpha x - \beta y = 1 \\ \alpha y + \beta x = 0 \end{array}\right\} \qquad \begin{array}{l} \alpha x - \beta y = 1 \\ \beta x + \alpha y = 0 \end{array}$

$D = \begin{vmatrix} \alpha & -\beta \\ \beta & \alpha \end{vmatrix} = \alpha^2 + \beta^2 \qquad D_x = \begin{vmatrix} 1 & -\beta \\ 0 & \alpha \end{vmatrix} = \alpha.$

$$D = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2 \qquad D_x = \begin{vmatrix} 1 & -b \\ 0 & a \end{vmatrix} = a.$$

$$D_y = \begin{vmatrix} a & 1 \\ b & 0 \end{vmatrix} = -b$$

$$x = \frac{D_x}{D} \quad , y = \frac{D_y}{D}$$

$$x = \frac{a}{a^2+b^2} \quad , y = \frac{-b}{a^2+b^2}$$

$z = a + bi$
$a, b \in \mathbb{R}$ είναι γνωστοί αριθμοί

$$\frac{1}{z} = x + yi =$$

$$= \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i$$

Διαίρεση μιγαδικών

$z_1 = a + bi \qquad z_2 = \gamma + \delta i \qquad \frac{1}{z_2} = \frac{\gamma}{\gamma^2+\delta^2} + \frac{-\delta}{\gamma^2+\delta^2} i$

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = (a+bi)\cdot\left(\frac{\gamma}{\gamma^2+\delta^2} + \frac{-\delta}{\gamma^2+\delta^2} i\right) =$$

$$= \frac{a\gamma}{\gamma^2+\delta^2} + \frac{-a\delta}{\gamma^2+\delta^2} i + \frac{b\gamma}{\gamma^2+\delta^2} i + \frac{-b\delta}{\gamma^2+\delta^2} \cdot i^2 =$$

$i^2 = -1$

$$= \frac{a\gamma}{\gamma^2+\delta^2} - \frac{a\delta}{\gamma^2+\delta^2} i + \frac{b\gamma}{\gamma^2+\delta^2} i + \frac{b\delta}{\gamma^2+\delta^2} =$$

$$= \frac{a\gamma + b\delta}{\gamma^2+\delta^2} + \frac{b\delta - a\delta}{\gamma^2+\delta^2} i$$

Δυνάμεις

$z^0 = 1$

$z^0 = 1$

$$z^\nu = \begin{cases} z & \nu = 1 \\ z^{\nu-1}, & \nu \geq 2 \end{cases} \qquad z^{-\nu} = \frac{1}{z^\nu} \quad z \neq 0 \ \nu \in \mathbb{N}$$

$i^2 = -1$

$i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$

$$i^k = i^{4\lambda+\nu} = (i^4)^\lambda \cdot i^\nu = i^\nu = \begin{cases} 1, & \nu = 0 \\ i, & \nu = 1 \\ -1, & \nu = 2 \\ -i, & \nu = 3 \end{cases}$$

Τετραγωνική ρίζα μιγαδικού

Κάθε μιγαδικός αριθμός έχει ακριβώς 2 τετραγωνικές ρίζες ($w = \alpha + \beta i$)

$w = z_0^2$ ή $w = z_1^2$

$\beta = 0$

$\alpha > 0 \qquad z_0 = \sqrt{\alpha} \qquad z_1 = -\sqrt{\alpha}$

$\alpha = 0 \qquad z_0 = 0$

$\alpha < 0 \qquad z_0 = i\sqrt{-\alpha} \qquad z_1 = -i\sqrt{-\alpha}$

$\beta \neq 0 \qquad z_0$ ή z_1

Να βρεθούν οι τετρ. ρίζες του $w = 3 - 4i$

$z = \alpha + \beta i$

$w = z^2 \Leftrightarrow 3 - 4i = (\alpha + \beta i)^2 \Leftrightarrow$

$3 - 4i = \alpha^2 + 2\alpha\beta i + \beta^2 i^2 \Leftrightarrow$

$3 - 4i = \alpha^2 + 2\alpha\beta i - \beta^2 \Leftrightarrow$

$$\begin{cases} \alpha^2 - \beta^2 = 3 \\ 2\alpha\beta = -4 \end{cases} \qquad (+4\alpha^2\beta^2)$$

$(\alpha^2 - \beta^2)^2 = 3^2 \Leftrightarrow \alpha^4 + \beta^4 - 2\alpha^2\beta^2 = 9 \Leftrightarrow \qquad (4\alpha^2\beta^2 = 16)$

$$a^2-b^2=3 \Leftrightarrow a^2+b^2 = 2ab \quad (4a^2b^2=16)$$
$$a^4+b^4+2a^2b^2=9+4a^2b^2 \Leftrightarrow$$
$$(a^2+b^2)^2=9+16=25 \Leftrightarrow$$
$$a^2+b^2=5$$

$$\left.\begin{array}{l} a^2-b^2=3 \\ a^2+b^2=5 \end{array}\right\} \quad 2a^2=8 \Leftrightarrow a^2=4 \quad b^2=1 \quad \left.\right\} \quad a=\pm 2,\ b=\pm 1$$

$a=2$, $b=-1$, $z=2-i$

$a=-2$, $b=1$, $z=-2+i$

Συζυγείς μιγαδικοί αριθμοί

$z=a+bi \qquad \bar{z}=a-bi$

- $\bar{\bar{z}}=\overline{a-bi}=a-(-bi)=a+bi=z$
- $z+\bar{z}=(a+bi)+(a-bi)=2a=2\,\mathrm{Re}(z)$
- $z-\bar{z}=(a+bi)-(a-bi)=2bi=2\,\mathrm{Im}(z)$
- $z\cdot\bar{z}=(a+bi)(a-bi)=a^2-\cancel{abi}+\cancel{abi}+b^2$
  $=a^2+b^2$

- $z=a+bi$ είναι πραγματικός $\Leftrightarrow$ $\bar{z}=z$

  $z=a+bi\in\mathbb{R}$ $\quad \mathrm{Im}(z)=0=b$

  $z-\bar{z}=2bi \Leftrightarrow \dfrac{z-\bar{z}}{2i}=0 \Leftrightarrow \bar{z}=z$

- $z=a+bi$ είναι φανταστικός $\Leftrightarrow$ $\bar{z}=-z$

  $z=a+bi$ φαντ. $\quad \mathrm{Re}(z)=0 \Leftrightarrow$

  $z+\bar{z}=0 \Leftrightarrow z=-\bar{z}$

$\overline{z+w}=\bar{z}+\bar{w}$

- $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \qquad z_2 \neq 0$
- $\overline{z_1+z_2+\dots+z_\nu} = \bar{z}_1+\bar{z}_2+\dots+\bar{z}_\nu$
- $\overline{z_1 \cdot z_2 \cdots z_\nu} = \bar{z}_1 \cdot \bar{z}_2 \cdots \bar{z}_\nu$
- $z_1 = z_2 = \dots = z_\nu = z$

$$\overline{\nu \cdot z} = \nu \cdot \bar{z}$$

$$\overline{(z^\nu)} = (\bar{z})^\nu$$

$$\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}} \quad , z \neq 0$$

Άσκηση

Να γραφεί στη μορφή $\alpha+\beta i$

$z_1 = (1+2i)(1-i) + (2+i)^2$

$= (1 - i + 2i - 2 \cdot i^2) + (2^2 + 2i2 + i^2) =$

$= (\cancel{1} - i + 2i + 2 + 4 + 4i - \cancel{1} =$

$= 6 + 5i$

$z_2 = \frac{1+i}{1+3i}$

$$z_2 = \frac{\quad}{1+3i}$$

$$= \frac{(1+i)(1-3i)}{(1+3i)(1-3i)} = \frac{1-3i+i-3i^2}{1^2+3^2} =$$

$$= \frac{4-2i}{10} = \frac{4}{10} - \frac{2}{10}i$$

$$z_3 = (1+i)^3 + \frac{1+2i}{1-i}$$

$$(1+i)^3 = (1+i)^2(1+i) =$$

$$= (1+2i+i^2)(1+i) =$$

$$= (\cancel{1}+2i-\cancel{1})(1+i) =$$

$$= 2i + 2i^2 = -2+2i$$

$$(1+i)^3 = 1 + 3\cdot 1\, i + 3i^2 + i^3$$

$$= 1 + 3i - 3 - i$$

$$= -2 + 2i$$

$$\frac{1+2i}{1-i} = \frac{(1+2i)(1+i)}{(1-i)(1+i)} = \frac{1+i+2i+2i^2}{1-i^2} =$$

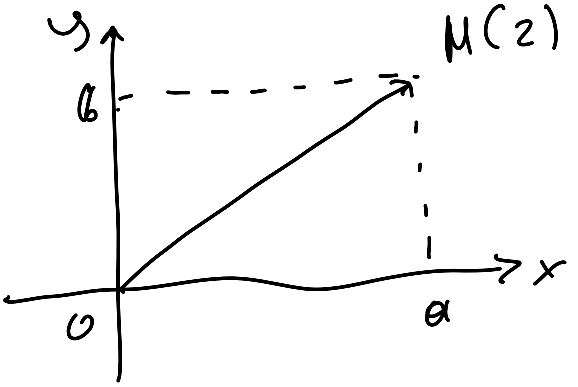
$$= \frac{1-2+3i}{1+1} = \frac{-1+3i}{2}$$

$$= -2+2i - \frac{1}{2} + \frac{3}{2}i =$$

$$= -\frac{4}{2} + \frac{4}{2}i - \frac{1}{2} + \frac{3}{2}i =$$

$$= -\frac{5}{2} + \frac{7}{2}i$$

- Μέτρο μιγαδικού αριθμού



$z = \alpha + \beta i$

$|z| = |\overrightarrow{OM}| = \sqrt{\alpha^2 + \beta^2}$

- $|z| = |\bar{z}| = |-z|$
- $|z| = 0 \Leftrightarrow z = 0$
- $\frac{1}{|z|} = \left|\frac{1}{z}\right| \quad z \neq 0$
- $|z|^2 = z \cdot \bar{z}$

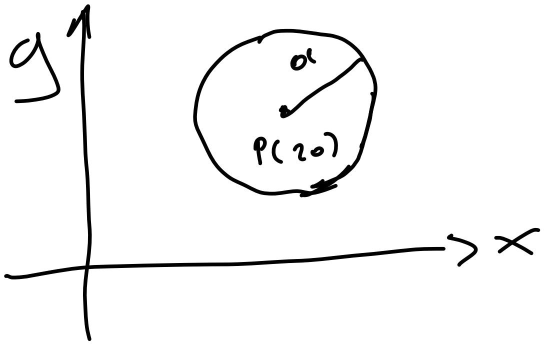
Αν $z \in \mathbb{R}$ $\quad |z|^2 = z^2$

z φανταστ. $\quad |z|^2 = -z^2$

- $|z_1 + z_2 + \dots + z_\nu| \leq |z_1| + |z_2| + \dots + |z_\nu|$
- $|z_1 \cdot z_2 \cdots z_\nu| = |z_1| \cdot |z_2| \cdots |z_\nu|$

$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

- $|z - z_0| = \alpha$



Ασκηση

Να γραφεί στη μορφή $\alpha + \beta i$

$$z_2 = \frac{z+i}{z-i} \quad , \text{με } z = x+yi \ , x,y\in\mathbb{R} \ z \neq i$$

$$z_2 = \frac{x+yi+i}{x+yi-i} = \frac{x+(y+1)i}{x+(y-1)i} =$$

$$= \frac{(x+(y+1)i)(x-(y-1)i)}{(x+(y-1)i)(x-(y-1)i)} =$$

$$= \frac{x^2+y^2-1}{x^2+(y-1)^2} + \frac{2x}{x^2+(y-1)^2} i$$

Ασκηση

Αν $\nu\in\mathbb{N}$ να βρεθούν οι δυνατές τιμές των παραστάσεων

i) $A = (1+i^{\nu})(1+i^{2\nu})$

- $\nu = 4\kappa + 0 \quad \kappa\in\mathbb{N} \quad i^{\nu} = i^{4\kappa} = 1$

$A = (1+1)(1+(i^{\nu})^2) = 2\cdot 2 = 4$

$i^{4\kappa} \, i^{1} = i$

$A = (1+1)(1+(1^2)) = 2 \cdot 2 \cdots$

- $\nu = 4k+1 \qquad i^\nu = i^{4k} \cdot i^1 = i$

$$A = (1+i)(1+i^2) = (1+i)(\cancelto{0}{1-1}) = 0$$

- $\nu = 4k+2 \qquad i^\nu = i^{4k} \cdot i^2 = -1$

$$A = (1+i^2)(1+(i^2)^2) = (\cancelto{0}{1-1})(1+i^4) = 0$$

- $\nu = 4k+3 \qquad i^\nu = i^{4k} \cdot i^3 = -i$

$$A = (1+i^3)(1+(i^3)^2) =$$

$$= (1-i)(\cancelto{0}{1-1}) = 0$$

$$(-i)^2 = (-1)^2 \cdot i^2 = 1 \cdot (-1) = -1$$

$$B = (1+i^\nu)(1+i^{2\nu})(1+i^{3\nu})\cdots(1+i^{\nu^2})$$

$i^\nu = x$

$$B = (1+x)(1+x^2)(1+x^3)\cdots(1+x^\nu)$$

$$i = \{1, i, -i, -1\}$$

$x = 1$ ------ $B = (1+1)(1+1)\cdots(1+1) =$

$$= \underbrace{2 \cdot 2 \cdots 2}_{\nu} = 2^\nu$$

$x = i$ --

$x = -i$ ..

$x = -1$

Άσκηση

Αν $x, y, z \in \mathbb{C}$

$$|x+y+z| \leq |x| + |y| + |z|$$

$$|x+y+z| = |(x+y) + z| \leq |x+y| + |z| \leq$$

$$\leq |x| + |y| + |z|$$

$$i^{\nu} = x$$

$$i^{2\nu} = i^{\nu+\nu} = i^{\nu} \cdot i^{\nu} = x \cdot x = x^2$$

$$i^{3\nu} = i^{\nu+\nu+\nu} = i^{\nu} \cdot i^{\nu} \cdot i^{\nu} = x\,x\,x = x^3$$

$$i^{\nu^2} = i^{\nu \cdot \nu} = i^{\underbrace{\nu+\nu+\dots+\nu}_{\nu \text{ φορές}}} =$$

$$= \underbrace{i^{\nu} \cdot i^{\nu} \cdot \ldots \cdot i^{\nu}}_{\nu \text{ φορές}} = \underbrace{x \cdot x \cdot x \cdots x}_{\nu \text{ φορές}} = x^{\nu}$$

Να υπολογίσετε το μέτρο των μιγαδικών

$z = \alpha + \beta i$

$|z| = \sqrt{\alpha^2 + \beta^2}$

$$z_1 = (1-i)^3 - (2+3i)^2$$

$$= (1^3 - 3 \cdot 1^2 \cdot i + 3 \cdot 1 \cdot i^2 - i^3) - (2^2 + 2 \cdot 2 \cdot 3i + (3i)^2) =$$

$$= (1 - 3i - 3 + i) - (4 + 12i - 9) =$$

$$= -2 - 2i - 4 - 12i + 9 = 3 - 14i$$

$$|z_1| = \sqrt{3^2 + (-14)^2} = \sqrt{9 + 196} = \sqrt{205}$$

$$|z_2| = \left|\frac{1+i}{\sqrt{3-i}}\right| = \frac{|1+i|}{|\sqrt{3-i}|} = \frac{2^{1/2}}{10^{1/4}} = \frac{2^{1/2}}{2^{1/4} \cdot 5^{1/4}} = \frac{2^{1/2-1/4}}{5^{1/4}} = \frac{2^{1/4}}{5^{1/4}} = \left(\frac{2}{5}\right)^{1/4}$$

$\left|\sqrt{3-i}\right| = \left|\sqrt{3-i}\right| = 10^{1/4}$

Αρ. $|1+i| = \sqrt{1^2+1^2} = \sqrt{2} = 2^{1/2}$

$\left|\sqrt{3-i}\right| = \sqrt{|3-i|} = \sqrt{\sqrt{3^2+(-1)^2}} = \sqrt{\sqrt{10}} = 10^{1/4}$

$|z_3| = \left|\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\nu}\right| = \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right|^{\nu} = \left(\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)^{\nu} = \left(\sqrt{\frac{1}{4}+\frac{3}{4}}\right)^{\nu} = 1^{\nu}$

$z = a + bi \quad |z| = \sqrt{a^2+b^2}$

Αν $z \in \mathbb{C}$ δ.ο $|z+3| + |z+2| - |z| - |z+1| \leq 4$

$$\left.\begin{array}{l} |z+3| = |(z+1)+2| \leq |z+1| + 2 \\ |z+2| \leq |z| + |2| \leq |z| + 2 \end{array}\right\}$$

$|z+3| + |z+2| \leq \underbrace{|z+1|} + |z| + 4$

Αν $z_1, z_2 \in \mathbb{C}$

i) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$|z|^2 = \bar{z}z = z\bar{z}$

$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= z_1\cdot\bar{z}_1 + z_1\cdot\bar{z}_2 + z_2\cdot\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

ii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2$

ii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2$

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1 \cdot \bar{z}_1 - z_1\bar{z}_2 - z_2 \cdot \bar{z}_1 + z_2\bar{z}_2 =$$

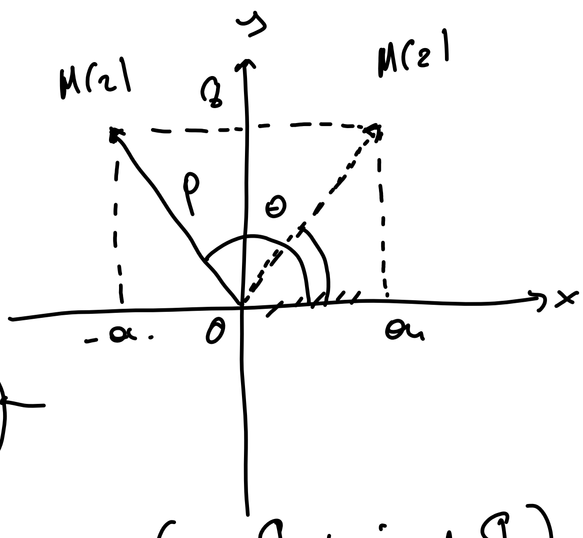
$$= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1$$

iii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

Από i) κ' ii)

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$$

Τριγωνομετρική μορφή μιγαδικού



$\cos\theta = \text{συν}\theta = \frac{\alpha}{\rho}$

$\sin\theta = \text{ημ}\theta = \frac{\beta}{\rho}$

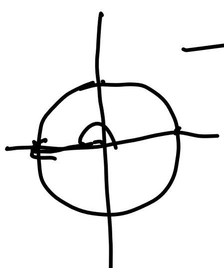
$\alpha = \rho\,\text{συν}\theta$

$\beta = \rho\,\text{ημ}\theta$

$z = \alpha + \beta i =$

$= \rho\,\text{συν}\theta + i\rho\,\text{ημ}\theta$

$= \rho(\text{συν}\theta + i\,\text{ημ}\theta)$



$$z = \rho(\text{συν}\theta + i\,\text{ημ}\theta) \qquad \rho = \sqrt{\alpha^2 + \beta^2}$$

$$\text{Arg}(z) = \theta$$

- $\text{Arg}(z) = 0 \Leftrightarrow z$ είναι θετικός πραγμ.
  $\theta = 0 \Rightarrow \text{ημ}\theta = 0 \qquad z = \rho(\text{συν}\theta) =$
- $\text{Arg}(z) = \pi \Leftrightarrow z$ είναι αρνητικός πραγμ.
  $\theta = \pi$
- $\text{Arg}(z) = \pi/2 \Rightarrow z$ είναι της μορφής $z = \beta i \quad \beta > 0$
  $z = \beta i \quad \beta < 0$
- $\text{Arg}(z) = \frac{3\pi}{2} \Rightarrow z$ $+/-$

- $Arg(z)=\frac{\pi}{2} \Rightarrow z$ ...

- $Arg(z)=\frac{3\pi}{2} \Rightarrow z$ $+/-$ $z=\alpha i$ $\alpha<0$

$$z_1=\rho_1(\sigma\upsilon\nu\theta_1+i\eta\mu\theta_1) \quad z_2=\rho_2(\sigma\upsilon\nu\theta_2+i\eta\mu\theta_2)$$

$$z_1=z_2 \iff \rho_1=\rho_2 \quad \theta_1=\theta_2 \text{ ή } \theta_1-\theta_2=2\kappa\pi$$

πολ/σμός

$$z_1=\rho_1(\sigma\upsilon\nu\theta_1+i\eta\mu\theta_1) \quad z_2=\rho_2(\sigma\upsilon\nu\theta_2+i\eta\mu\theta_2)$$

$$z_1\cdot z_2=\rho_1(\sigma\upsilon\nu\theta_1+i\eta\mu\theta_1)\cdot\rho_2(\sigma\upsilon\nu\theta_2+i\eta\mu\theta_2)=$$

$$=\rho_1\rho_2\left[\sigma\upsilon\nu\theta_1\cdot\sigma\upsilon\nu\theta_2+i\sigma\upsilon\nu\theta_1\eta\mu\theta_2+i\eta\mu\theta_1\cdot\sigma\upsilon\nu\theta_2+i^2\eta\mu\theta_1\eta\mu\theta_2\right]=$$

$$=\rho_1\rho_2\left[\sigma\upsilon\nu\theta_1\sigma\upsilon\nu\theta_2-\eta\mu\theta_1\eta\mu\theta_2+i(\sigma\upsilon\nu\theta_1\eta\mu\theta_2+\eta\mu\theta_1\sigma\upsilon\nu\theta_2)\right]$$

$$=\rho_1\rho_2\left[\sigma\upsilon\nu(\theta_1+\theta_2)+i\eta\mu(\theta_1+\theta_2)\right]$$

$$\frac{z_1}{z_2}=\frac{\rho_1}{\rho_2}\left[\sigma\upsilon\nu(\theta_1-\theta_2)+i\eta\mu(\theta_1-\theta_2)\right]$$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
συν	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
ημ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
εφ.	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$	0	$-$	0

- $\eta\mu(\frac{\pi}{2}-\theta)=\sigma\upsilon\nu\theta$ $\quad\eta\mu(\pi-\theta)=\eta\mu\theta$ $\quad\eta\mu(\frac{\pi}{2}+\theta)=\sigma\upsilon\nu\theta$
- $\sigma\upsilon\nu(\frac{\pi}{2}-\theta)=\eta\mu\theta$ $\quad\sigma\upsilon\nu(\pi-\theta)=-\sigma\upsilon\nu\theta$ $\quad\sigma\upsilon\nu(\frac{\pi}{2}+\theta)=-\eta\mu\theta$
- $\varepsilon\varphi(\frac{\pi}{2}-\theta)=\sigma\varphi\theta$ $\quad\varepsilon\varphi(\pi-\theta)=-\varepsilon\varphi\theta$ $\quad\varepsilon\varphi(\frac{\pi}{2}+\theta)=-\sigma\varphi\theta$
- $\eta\mu(2\pi-\theta)=-\eta\mu\theta$ $\quad\eta\mu(\pi+\theta)=-\eta\mu\theta$ $\quad\eta\mu(-\theta)=-\eta\mu\theta$
- $\sigma\upsilon\nu(2\pi-\theta)=\sigma\upsilon\nu\theta$ $\quad\sigma\upsilon\nu(\pi+\theta)=-\sigma\upsilon\nu\theta$ $\quad\sigma\upsilon\nu(-\theta)=\sigma\upsilon\nu\theta$

- $\text{ημ}(2\pi-\theta) = -\text{ημ}\theta$ $\quad$ $\text{ημ}(\pi+\theta) = -\text{ημ}\theta$ $\quad$ $\text{ημ}(-\theta) = -\text{ημ}\theta$
- $\text{συν}(2\pi-\theta) = \text{συν}\theta$ $\quad$ $\text{συν}(\pi+\theta) = -\text{συν}\theta$ $\quad$ $\text{συν}(-\theta) = \text{συν}\theta$
- $\text{εφ}(2\pi-\theta) = -\text{εφ}\theta$ $\quad$ $\text{εφ}(\pi+\theta) = \text{εφ}\theta$ $\quad$ $\text{εφ}(-\theta) = -\text{εφ}\theta$

- $\text{ημ}^2\theta + \text{συν}^2\theta = 1$ $\quad$ $\frac{1}{\text{συν}^2\theta} = 1 + \text{εφ}^2\theta$

- $\text{ημ}(\alpha \pm \beta) = \text{ημ}\alpha\,\text{συν}\beta \pm \text{συν}\alpha \cdot \text{ημ}\beta$
- $\text{συν}(\alpha \pm \beta) = \text{συν}\alpha\,\text{συν}\beta \mp \text{ημ}\alpha \cdot \text{ημ}\beta$
- $\text{εφ}(\alpha \pm \beta) = \frac{\text{εφ}\alpha \pm \text{εφ}\beta}{1 - \text{εφ}\alpha\,\text{εφ}\beta}$

- $\text{ημ}2\theta = 2\text{ημ}\theta\,\text{συν}\theta$
- $\text{συν}2\theta = \text{συν}^2\theta - \text{ημ}^2\theta = 2\text{συν}^2\theta - 1 = 1 - 2\text{ημ}^2\theta$
- $\text{συν}^2\theta = \frac{1+\text{συν}2\theta}{2}$
- $\text{ημ}^2\theta = \frac{1-\text{συν}2\theta}{2}$
- $\text{εφ}2\theta = \frac{2\text{εφ}\theta}{1+\text{εφ}^2\theta}$

- $\text{ημ}\alpha + \text{ημ}\beta = 2\text{ημ}\frac{\alpha+\beta}{2}\,\text{συν}\frac{\alpha-\beta}{2}$
- $\text{ημ}\alpha - \text{ημ}\beta = 2\text{ημ}\frac{\alpha-\beta}{2}\,\text{συν}\frac{\alpha+\beta}{2}$
- $\text{συν}\alpha + \text{συν}\beta = 2\text{συν}\frac{\alpha+\beta}{2} \cdot \text{συν}\frac{\alpha-\beta}{2}$
- $\text{συν}\alpha - \text{συν}\beta = 2\text{ημ}\frac{\alpha+\beta}{2}\,\text{ημ}\frac{\alpha-\beta}{2}$
- $\text{ημ}\alpha\,\text{ημ}\beta = \frac{1}{2}[\text{συν}(\alpha-\beta) - \text{συν}(\alpha+\beta)]$
- $\text{ημ}\alpha\,\text{συν}\beta = \frac{1}{2}[\text{ημ}(\alpha+\beta) + \text{ημ}(\alpha-\beta)]$
- $\text{συν}\alpha \cdot \text{συν}\beta = \frac{1}{2}[\text{συν}(\alpha+\beta) + \text{συν}(\alpha-\beta)]$

Αν $z_1, z_2, \dots, z_\nu$ ν μιγαδικοί $\nu \geq 2$ $\theta_1, \theta_2, \dots, \theta_\nu$

$z_1 \cdot z_2 \cdots z_\nu$ έχει μέτρο $|z_1 \cdot z_2 \cdots z_\nu| = |z_1| \cdot |z_2| \cdots |z_\nu|$

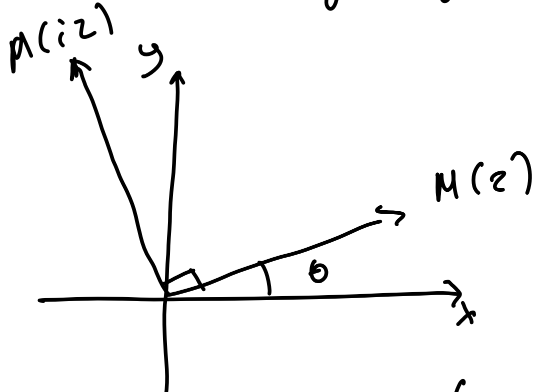
$\theta_1+\theta_2+\dots+\theta_\nu$ $\qquad$ $z=1=1+0i$

Αν $z=\rho(\sigma\upsilon\nu\theta+i\eta\mu\theta)$ $\qquad$ $1=1(\sigma\upsilon\nu 0+i\eta\mu 0)$

$$\frac{1}{z}=\frac{1(\sigma\upsilon\nu 0+i\eta\mu 0)}{\rho(\sigma\upsilon\nu\theta+i\eta\mu\theta)}=\frac{1}{\rho}\left(\sigma\upsilon\nu(0-\theta)+i\eta\mu(0-\theta)\right)$$

Αν $z=i$ $\qquad$ $i=1\left(\sigma\upsilon\nu\frac{\pi}{2}+i\eta\mu\frac{\pi}{2}\right)$

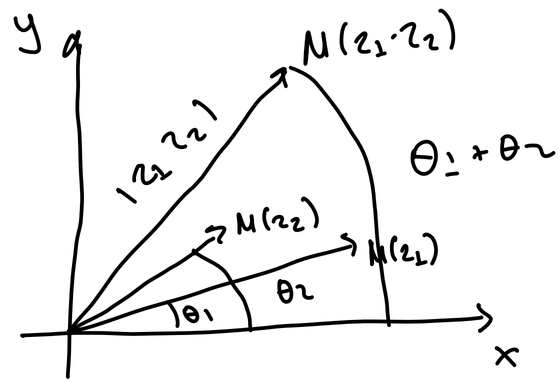
$i\cdot z \to$ αλλαγή γωνίας κατά $\pi/2$



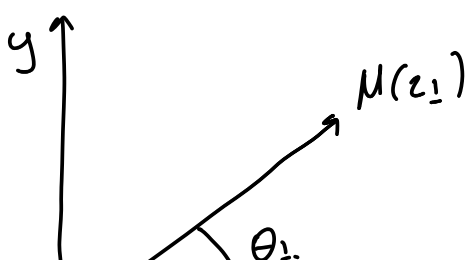
Πολλαπλασιασμός - $z_1=\rho_1(\sigma\upsilon\nu\theta_1+i\eta\mu\theta_1)$

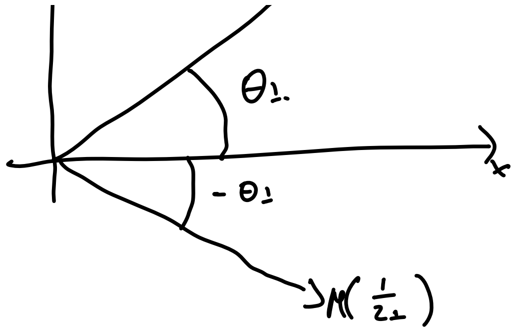
$z_2=\rho_2(\sigma\upsilon\nu\theta_2+i\eta\mu\theta_2)$

$$z_1\cdot z_2=\rho_1\rho_2\left[\sigma\upsilon\nu(\theta_1+\theta_2)+i\eta\mu(\theta_1+\theta_2)\right]$$



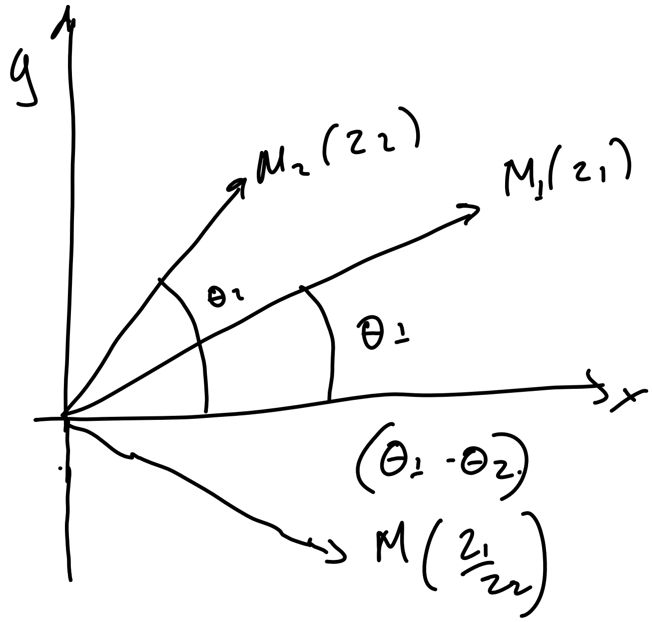
$$\frac{1}{z}=\frac{1}{z_1}$$





Πηλίκο

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}\left(συν(\theta_1-\theta_2) + iημ(\theta_1-\theta_2)\right)$$



Θεώρημα de Moivre

Αν $z = \rho(συνθ + iημθ)$ ν είναι θετικός ακέραιος

$$z^\nu = \rho^\nu\left(συν(\nu\theta) + iημ(\nu\theta)\right)$$

- $\nu=1$ $z^1 = \rho^1\left(συν(\theta) + iημ(\theta)\right)$
- Έστω ότι ισχύει για $\nu=k$ δηλ.

$$z^k = \rho^k\left(συν(k\theta) + iημ(k\theta)\right)$$

- Αρκεί ν.δ.ο ισχύει για $\nu=k+1$

$$z^{k+1} = \rho^{k+1}\left(συν((k+1)\theta) + iημ((k+1)\theta)\right)$$

- Αρκεί ν.δ.ο ισχύει για $\nu = k+1$

δηλ. $z^{k+1} = \rho^{k+1}\left(\text{συν}((k+1)\theta) + i\,\text{ημ}((k+1)\theta)\right)$

$$z^{k+1} = z^{k}\cdot z = \rho^{k}\left(\text{συν}(k\theta) + i\,\text{ημ}(k\theta)\right)\cdot \rho\left(\text{συν}(\theta) + i\,\text{ημ}(\theta)\right) =$$

$$= \rho^{k+1}\left(\text{συν}(k\theta+\theta) + i\,\text{ημ}(k\theta+\theta)\right) =$$

$$= \rho^{k+1}\left(\text{συν}((k+1)\theta) + i\,\text{ημ}((k+1)\theta)\right)$$

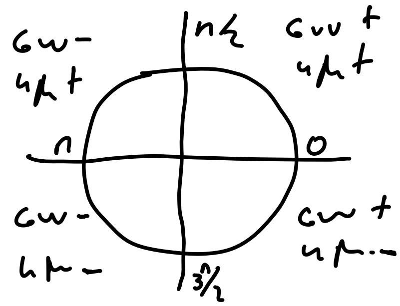
$$z^{-\nu} = \frac{1}{z^{\nu}} = \frac{1\left(\text{συν}(0) + i\,\text{ημ}(0)\right)}{\rho^{\nu}\left(\text{συν}(\nu\theta) + i\,\text{ημ}(\nu\theta)\right)} =$$

$$= \rho^{-\nu}\left(\text{συν}(-\nu\theta) + i\,\text{ημ}(-\nu\theta)\right)$$

Άσκηση 1 Να γραφούν σε τριγ., να βρεθεί το πρωτεύον όρισμα

$- z_1 = \underset{\alpha}{1} + i\underset{b}{\sqrt{3}}$ $\quad \rho = \sqrt{\alpha^2 + b^2} = \sqrt{1 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\left.\begin{aligned} \text{συν}\theta &= \frac{\alpha}{\rho} = \frac{1}{2} \\ \text{ημ}\theta &= \frac{b}{\rho} = \frac{\sqrt{3}}{2} \end{aligned}\right\} \quad \theta = \frac{\pi}{3} = Arg(z)$$



$$z_1 = 2\left(\text{συν}\left(\frac{\pi}{3}\right) + i\,\text{ημ}\left(\frac{\pi}{3}\right)\right)$$

$$\left.\begin{aligned} \text{συν} &\ + \\ \text{ημ} &\ + \end{aligned}\right\}$$ θ τόξο του $1^{ου}$ τεταρτημορίου

$- z_2 = -\underset{\alpha}{\sqrt{3}} + i\cdot \underset{b}{1}$

$$\rho = \sqrt{\alpha^2 + b^2} = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\begin{aligned} \text{συν}\theta &= \frac{\alpha}{\rho} = -\frac{\sqrt{3}}{2} \quad - \\ \text{ημ}\theta &= \frac{b}{\rho} = \frac{1}{2} \quad + \end{aligned} \qquad \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6} = Arg(z)$$

$$z_2 = 2\left(\text{συν}\left(\frac{5\pi}{6}\right) + i\,\text{ημ}\left(\frac{5\pi}{6}\right)\right)$$

$$\left.\begin{aligned} \text{συν} &\ - \\ \text{ημ} &\ + \end{aligned}\right\} \quad \theta = \pi - \frac{\pi}{6}$$

$- z_3 = -\underset{\alpha}{\frac{1}{2}} - \underset{b}{\frac{\sqrt{3}}{2}}\, i$

$$\rho = \sqrt{\alpha^2 + b^2} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

- $z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$$\rho = \sqrt{a^2+b^2} = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\sigma\upsilon\nu\theta = \frac{a}{\rho} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2} \qquad \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} = Arg(z)$$

$$\eta\mu\theta = \frac{b}{\rho} = \frac{-\frac{\sqrt{3}}{2}}{1} = -\frac{\sqrt{3}}{2}$$

$$\left.\begin{array}{l}\sigma\upsilon\nu\theta \ - \\ \eta\mu\theta \ -\end{array}\right\} \theta = \pi + \frac{\pi}{3} \qquad z_3 = 1\cdot\left(\sigma\upsilon\nu\left(\frac{4\pi}{3}\right) + i\eta\mu\left(\frac{4\pi}{3}\right)\right)$$

- $z_4 = \underset{a}{1} - i\cdot \underset{b}{1}$

$$\rho = \sqrt{a^2+b^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\sigma\upsilon\nu\theta = \frac{a}{\rho} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} = Arg(z_4)$$

$$\eta\mu\theta = \frac{b}{\rho} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\left.\begin{array}{l}\sigma\upsilon\nu\theta \ + \\ \eta\mu\theta \ -\end{array}\right\} \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$z_4 = \sqrt{2}\left(\sigma\upsilon\nu\left(\frac{7\pi}{4}\right) + i\eta\mu\left(\frac{7\pi}{4}\right)\right)$$

Άσκηση 2

Να βρεθούν οι τριγ. αρ. πρωτεύον όρισμα

- $z_1 = 3 \qquad z_1 = 3 + 0i \qquad \rho = 3$

$$\sigma\upsilon\nu\theta = \frac{a}{\rho} = \frac{3}{3} = 1 \qquad \theta = 0$$

$$\eta\mu\theta = \frac{b}{\rho} = \frac{0}{3} = 0 \qquad z_1 = 3(\sigma\upsilon\nu 0 + i\eta\mu 0)$$

κάθε θετικός αριθμός έχει μέτρο το ίδιο αριθμό κ' πρωτεύον το 0.

- $z_2 = -3 \qquad z_2 = -3 + 0i \qquad \rho = \sqrt{(-3)^2 + 0^2} = 3$

$$\sigma\upsilon\nu\theta = \frac{a}{\rho} = \frac{-3}{3} = -1 \qquad \theta = \pi = Arg(z_2)$$

$\sigma\upsilon\nu\theta = \frac{\alpha}{\rho} = \frac{-3}{3} = -1$ $\quad \theta = \pi = Arg(z_2)$

$\eta\mu\theta = \frac{\beta}{\rho} = \frac{0}{3} = 0$

$$z_2 = 3(\sigma\upsilon\nu(\pi) + i\eta\mu(\pi))$$

Κάθε αρνητικός αριθμός έχει μέτρο την απόλυτη τιμή του κ' πρωτεύον όρισμα το π.

$- z_3 = 2i \quad \rho = \sqrt{0^2 + 2^2} = 2$

$\sigma\upsilon\nu\theta = \frac{\alpha}{\rho} = \frac{0}{2} = 0 \quad \theta = \frac{\pi}{2} = Arg(z_3)$

$\eta\mu\theta = \frac{\beta}{\rho} = \frac{2}{2} = 1 \quad z_3 = 2\left(\sigma\upsilon\nu\left(\frac{\pi}{2}\right) + i\eta\mu\left(\frac{\pi}{2}\right)\right)$

Κάθε φανταστικός αριθ. με συντελεστή θετικό έχει μέτρο τον συντελεστή κ' πρωτεύον όρισμα την γων. π/2

$- z_3 = -3i \quad \rho = \sqrt{0^2 + (-3)^2} = 3$

$\sigma\upsilon\nu\theta = \frac{\alpha}{\rho} = \frac{0}{3} = 0 \quad \theta = \frac{3\pi}{2} = Arg(z_4)$

$\eta\mu\theta = \frac{\beta}{\rho} = \frac{-3}{3} = -1$

$$z_4 = 3\left(\sigma\upsilon\nu\left(\frac{3\pi}{2}\right) + i\eta\mu\left(\frac{3\pi}{2}\right)\right)$$

Κάθε φανταστικός αριθ. με συντελεστή αρνητικό έχει μέτρο την απόλυτη τιμή του συντελεστή και πρωτεύον όρ. την γων. 3π/2

Άσκηση 3 Να γραφεί σε τριγωνομετρική

$z_1 = \frac{1+i}{\sqrt{3}+i}$ $\quad z = \alpha + \beta i \quad \rho = \sqrt{\alpha^2 + \beta^2}$

$\rho = \sqrt{1^2 + 1^2} = \sqrt{2} \quad \sigma\upsilon\nu\theta = \frac{\alpha}{\rho}, \eta\mu\theta = \frac{\beta}{\rho}$

Αριθμητής : $1 + i$

Αριθμητής : $1+i$ $\qquad \rho = \sqrt{1+1} = \sqrt{2}$

$\sigma\upsilon\nu\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt{2}}{2} \quad + \qquad \theta = \frac{\pi}{4}$

$\eta\mu\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad +$

$1+i = \sqrt{2}\left(\sigma\upsilon\nu \frac{\pi}{4} + i\eta\mu \frac{\pi}{4}\right)$

Παρονομαστής : $\sqrt{3}+i$ $\qquad \rho = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$$\left.\begin{aligned} \sigma\upsilon\nu\theta &= \frac{\sqrt{3}}{2} \quad + \\ \eta\mu\theta &= \frac{1}{2} \quad + \end{aligned}\right\} \quad \frac{\pi}{6}$$

$\sqrt{3}+i = 2\left(\sigma\upsilon\nu \frac{\pi}{6} + i\eta\mu \frac{\pi}{6}\right)$

$$z_1 = \frac{\sqrt{2}\left(\sigma\upsilon\nu \frac{\pi}{4} + i\eta\mu \frac{\pi}{4}\right)}{2\left(\sigma\upsilon\nu \frac{\pi}{6} + i\eta\mu \frac{\pi}{6}\right)} = \frac{\sqrt{2}}{2}\left(\sigma\upsilon\nu\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\eta\mu\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right) =$$

$$= \frac{\sqrt{2}}{2}\left(\sigma\upsilon\nu\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) + i\eta\mu\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right)\right) =$$

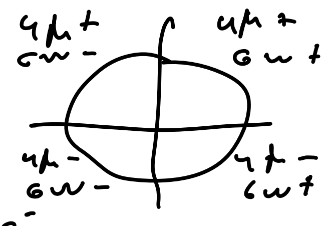
$$= \frac{\sqrt{2}}{2}\left(\sigma\upsilon\nu\left(\frac{\pi}{12}\right) + i\eta\mu\left(\frac{\pi}{12}\right)\right)$$

$z_2 = \frac{5 + 11\sqrt{3}i}{7 - 4\sqrt{3}i} \qquad \rho = \sqrt{a^2 + b^2} \quad \sigma\upsilon\nu\theta = \frac{a}{\rho}, \ \eta\mu\theta \ \frac{b}{\rho}$

αρ. $5+11\sqrt{3}i \qquad \rho = \sqrt{5^2 + (11\sqrt{3})^2} = \sqrt{25 + 363} = \sqrt{388}$

$$z_2 = \frac{(5 + 11\sqrt{3}i)(7 + 4\sqrt{3}i)}{(7 - 4\sqrt{3}i)(7 + 4\sqrt{3}i)} = \frac{35 + 20\sqrt{3}i + 77\sqrt{3}i - 132}{49 + 48} =$$

$$= \frac{-97 + 97\sqrt{3}i}{97} = -1 + \sqrt{3}i$$



$\rho = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$ $\qquad \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$$\left.\begin{aligned} \sigma\upsilon\nu\theta &= \frac{a}{\rho} = \frac{-1}{2} = -\frac{1}{2} \\ \eta\mu\theta &= \frac{b}{\rho} = \frac{\sqrt{3}}{2} \end{aligned}\right\} \quad \frac{\pi}{3}$$

$$z_2 = 2\left(\sigma\upsilon\nu\left(\frac{2\pi}{3}\right) + i\eta\mu\left(\frac{2\pi}{3}\right)\right)$$

Άσκηση 4

$z_1 = \dfrac{\sqrt{6} - i\sqrt{2}}{2}$ $\qquad \rho = \sqrt{\alpha^2 + \beta^2} \qquad \sigma\upsilon\nu\theta = \dfrac{\alpha}{\rho} \qquad \eta\mu\theta = \dfrac{\beta}{\rho}$

$$z_1 = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i$$

$$\rho = \sqrt{\left(\frac{\sqrt{6}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{6}{4} + \frac{2}{4}} = \sqrt{\frac{8}{4}} = \sqrt{2}$$

$$\left.\begin{aligned} \sigma\upsilon\nu\theta &= \frac{\alpha}{\rho} = \frac{\frac{\sqrt{6}}{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{2\cdot 3}}{2\sqrt{2}} = \frac{\cancel{\sqrt{2}}\cdot\sqrt{3}}{2\cancel{\sqrt{2}}} = \frac{\sqrt{3}}{2} \\ \eta\mu\theta &= \frac{\beta}{\rho} = \frac{-\frac{\sqrt{2}}{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2\sqrt{2}} = -\frac{1}{2} \end{aligned}\right\} \quad \frac{\pi}{6}$$

$$2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

$$z_1 = \sqrt{2}\left(\sigma\upsilon\nu\left(\frac{11\pi}{6}\right) + i\,\eta\mu\left(\frac{11\pi}{6}\right)\right)$$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$	2π
$\sigma\upsilon\nu\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1	0	1
$\eta\mu\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1	0

$z_2 = 1 - i \qquad \rho = \sqrt{\alpha^2 + \beta^2} \qquad \sigma\upsilon\nu\theta = \dfrac{\alpha}{\rho} \quad \eta\mu\theta = \dfrac{\beta}{\rho}$

$$\rho = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\sigma\upsilon\nu\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad +$$

$$\eta\mu\theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \quad -$$

$$2\pi - \frac{\pi}{4} = \frac{7\pi}{4} = Arg(z_2)$$

$$z_2 = \sqrt{2}\left(\sigma\upsilon\nu\left(\frac{7\pi}{4}\right) + i\,\eta\mu\left(\frac{7\pi}{4}\right)\right)$$

Άσκηση 5

$\sqrt{2 \cdot \alpha^2} = \sqrt{(\sigma\upsilon\nu\theta)^2 + (\sigma\upsilon\nu\eta\mu\theta)^2} =$

Άσκηση 5

$z_1 = \sigma\upsilon\nu\theta - i\eta\mu\theta$

$\rho = \sqrt{\alpha^2+\beta^2} = \sqrt{(\sigma\upsilon\nu\theta)^2 + (-\eta\mu\theta)^2} = \sqrt{\sigma\upsilon\nu^2\theta + \eta\mu^2\theta} = \sqrt{1} = 1$

$\sigma\upsilon\nu\theta = \frac{\alpha}{\rho} = \frac{\sigma\upsilon\nu\theta}{1} = \sigma\upsilon\nu\theta$

$\eta\mu\theta = \frac{\beta}{\rho} = \frac{-\eta\mu\theta}{1} = -\eta\mu\theta = \eta\mu(-\theta)$ $\quad 2\pi-\theta$

$z_1 = 1(\sigma\upsilon\nu(2\pi-\theta) + i\eta\mu(2\pi-\theta))$

$z_2 = -\sigma\upsilon\nu\theta + i\eta\mu\theta$ $\quad \rho = 1$

$\sigma\upsilon\nu\theta = \frac{\alpha}{\rho} = \frac{-\sigma\upsilon\nu\theta}{1} = -\sigma\upsilon\nu\theta$ $\quad (\pi-\theta)$

$\eta\mu\theta = \frac{\beta}{\rho} = \frac{\eta\mu\theta}{1} = \eta\mu\theta$

$z_2 = 1(\sigma\upsilon\nu(\pi-\theta) + i\eta\mu(\pi-\theta))$

$z_3 = -\sigma\upsilon\nu\theta - i\eta\mu\theta$ $\quad \rho = 1$

$\sigma\upsilon\nu\theta = \frac{-\sigma\upsilon\nu\theta}{1} = -\sigma\upsilon\nu\theta$ $\quad (\pi+\theta)$

$\eta\mu\theta = \frac{-\eta\mu\theta}{1} = -\eta\mu\theta$

$z_3 = 1\cdot(\sigma\upsilon\nu(\pi+\theta) + i\eta\mu(\pi+\theta))$

$z_4 = \eta\mu\theta + i\sigma\upsilon\nu\theta$ $\quad \rho = 1$

$\eta\mu\theta = \sigma\upsilon\nu(\frac{\pi}{2} - \theta)$

$\sigma\upsilon\nu\theta = \eta\mu(\frac{\pi}{2} - \theta)$

$z_4 = 1\cdot(\sigma\upsilon\nu(\frac{\pi}{2}-\theta) + i\eta\mu(\frac{\pi}{2}-\theta))$

Πολυωνυμικές εξισώσεις στο $\mathbb{C}$

$P(x) = a_\nu x^\nu + a_{\nu-1} x^{\nu-1} + a_{\nu-2} x^{\nu-2} + \dots + a_2 x^2 + a_1 x + a_0 \leftarrow \mathbb{R}$

$P(z) = a_\nu z^\nu + a_{\nu-1} z^{\nu-1} + a_{\nu-2} z^{\nu-2} + \dots + a_2 z^2 + a_1 z + a_0 \leftarrow \mathbb{C}$

z_0 είναι ρίζα του $P(z)$ $\quad P(z_0) = 0$

$a_\nu z_0^\nu + a_{\nu-1} z_0^{\nu-1} + \dots + a_1 z_0 + a_0 = 0$

- Κάθε πολυωνυμική έχει μια τουλάχιστο μιγαδική ρίζα

- Θεώρημα.
Κάθε πολυωνυμική εξίσωση $a_\nu z^\nu + a_{\nu-1} z^{\nu-1} + \dots + a_1 z + a_0 = 0$
ν ακριβώς ρίζες όχι κατ' ανάγκη διαφορετικές

- $a z^2 + \beta z + \gamma = 0 \qquad a, \beta, \gamma \in \mathbb{R}$

$z^2 + \frac{\beta}{a} z + \frac{\gamma}{a} = 0 \iff z^2 + 2\frac{\beta}{2a} z + \frac{\beta^2}{4a^2} - \frac{\beta^2}{4a^2} + \gamma/a = 0$

$\iff \left(z + \frac{\beta}{2a}\right)^2 = \frac{\beta^2 - 4a\gamma}{2a} \qquad \Delta = \beta^2 - 4a\gamma$

$z_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4a\gamma}}{2a} \qquad \Delta > 0$

$\Delta = 0$

$z_0 = \frac{-\beta}{2a}$

$z_{1,2} = \frac{-\beta \pm i\sqrt{|\Delta|}}{2a} \qquad \Delta < 0$

$d^2 = \beta^2 - 4a\gamma$

$\left(z + \frac{\beta}{2a}\right)^2 = \left(\frac{d}{2a}\right)^2$

$$z_1+z_2=-\frac{\beta}{\alpha} \qquad z_1\cdot z_2=\frac{\gamma}{\alpha}$$

- Θεώρημα

Αν ο αριθμός z_0 είναι ρίζα

$a_\nu x^\nu + a_{\nu-1} x^{\nu-1}+\dots+a_0=0$ τότε κ' αριθμός $\overline{z_0}$

είναι επίσης ρίζα

Άσκηση

$z^2+z+1=0 \qquad \Delta=\beta^2-4\alpha\gamma$

$\alpha z^2+\beta z+\gamma=0 \qquad =1-4\cdot1\cdot1=1-4=-3<0$

$$z_{1,2}=\frac{-\beta\pm i\sqrt{|\Delta|}}{2\alpha}=\frac{-1\pm i\sqrt{|3|}}{2\cdot1}=\frac{-1\pm i\sqrt{3}}{2}=$$

$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i \quad \text{κ'} \quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

Άσκηση $\lambda\in\mathbb{R}=?$: $1+i$ να είναι ρίζα της

$2z^2+\lambda z+4=0$

z είναι ρίζα

$\bar{z}$ είναι ρίζα

$z_1=1+i \quad \overline{z_1}=1-i$

$$z_1+z_2=-\frac{\beta}{\alpha} \qquad z_1\cdot z_2=\frac{\gamma}{\alpha}$$

$z_1+z_2=-\frac{\lambda}{\alpha}$

$(1+i)+(1-i)=-\frac{\lambda}{2} \Leftrightarrow$

$2=-\frac{\lambda}{2} \Leftrightarrow \lambda=-4$

Επίλυση της εξίσωσης $z^\nu=a \quad a\in\mathbb{C}, \nu\in\mathbb{N} \quad \nu\geq 2$

$a=\rho(\sigma\upsilon\nu\theta+i\eta\mu\theta)$

$$a=\rho(\sigma\upsilon\nu\theta+i\eta\mu\theta)$$

$$z_k=\sqrt[\nu]{\rho}\left(\sigma\upsilon\nu\frac{\theta+2k\pi}{\nu}+i\eta\mu\frac{\theta+2k\pi}{\nu}\right)$$

$$k=0,1,2,\dots,\nu-1$$

z_λ, $z_{\lambda+1}$ διαφέρουν στο όρισμα τους κατά $\frac{2\pi}{\nu}$

Άσκηση

Να λυθεί η εξίσωση $z^6+64=0$ κ' να παρασταθούν οι ρίζες.

$$z^6=-64 \qquad z^\nu=a \qquad a=\rho(\sigma\upsilon\nu\theta+i\eta\mu\theta)$$

$$-64=64(\sigma\upsilon\nu\pi+i\eta\mu\pi)$$

$$z_k=\sqrt[\nu]{\rho}\left(\sigma\upsilon\nu\frac{2k\pi+\theta}{\nu}+i\eta\mu\frac{2k\pi+\theta}{\nu}\right)$$

$$=\sqrt[6]{64}\left(\sigma\upsilon\nu\frac{2k\pi+\pi}{6}+i\eta\mu\frac{2k\pi+\pi}{6}\right)$$

$$k=0,1,2,\dots,5$$

$$\sqrt[6]{64}=\sqrt[6]{2^6}=2$$

$$k=0 \quad z_0=2\left(\sigma\upsilon\nu\left(\frac{\pi}{6}\right)+i\eta\mu\left(\frac{\pi}{6}\right)\right)$$

$$=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$$

$$k=1 \quad z_1=2\left(\sigma\upsilon\nu\frac{3\pi}{6}+i\eta\mu\frac{3\pi}{6}\right)$$

$$=2\left(\sigma\upsilon\nu\left(\frac{\pi}{2}\right)+i\eta\mu\frac{\pi}{2}\right)$$

$$=2(0+i1)$$

$$k=2 \quad z_2=2\left(\sigma\upsilon\nu\left(\frac{5\pi}{6}\right)+i\eta\mu\left(\frac{5\pi}{6}\right)\right)$$

$k=3 \quad z_3 = 2\left(\sigma v\nu\left(\frac{7\pi}{6}\right) + i\eta\mu\left(\frac{7\pi}{6}\right)\right)$

$k=4 \quad z_4 = 2\left(\sigma v\nu\left(\frac{9\pi}{6}\right) + i\eta\mu\left(\frac{9\pi}{6}\right)\right)$

$k=5 \quad z_5 = 2\left(\sigma v\nu\left(\frac{11\pi}{6}\right) + i\eta\mu\left(\frac{11\pi}{6}\right)\right)$

Άσκηση

Να λυθεί η εξίσωση

$z^{\nu-1} + z^{\nu-2} + \dots + z + 1 = 0 \qquad \nu \in \mathbb{N} \quad \nu \geq 2$

$(z^\nu - 1) = (z-1)(z^{\nu-1} + z^{\nu-2} + \dots + z + 1)$

$\frac{z^\nu - 1}{z-1} = z^{\nu-1} + z^{\nu-2} + \dots + z + 1 = 0 \iff$

$\frac{z^\nu - 1}{z-1} = 0 \quad z \neq 1 \quad \left.\right\} \begin{array}{l} z^\nu - 1 = 0 \\ z \neq 1 \end{array}$

$1 = 1 \cdot (\sigma v\nu 0 + i\eta\mu 0)$

$a = \rho(\sigma v\nu\theta + i\eta\mu\theta)$

$z^\nu = a$

$z_k = \sqrt[\nu]{\rho}\left(\sigma v\nu \frac{2k\pi+\theta}{\nu} + i\eta\mu \frac{2k\pi+\theta}{\nu}\right)$

$k = 0, 1, \dots, \nu-1$

$z_k = \sqrt[\nu]{1}\left(\sigma v\nu \frac{2k\pi+0}{\nu} + i\eta\mu \frac{2k\pi+0}{\nu}\right)$

$\left(\frac{2k\pi}{\cdot}\right) \dots \left(\frac{2k\pi}{\cdot}\right)$

$z_k = \sqrt[\nu]{1} \; (\text{συν} \ldots$

$$z_k = \text{συν}\left(\frac{2k\pi}{\nu}\right) + i\,\text{ημ}\left(\frac{2k\pi}{\nu}\right)$$

$$k = 0, 1, 2, \ldots, \nu - 1$$

$k=0$: $z_0 = \text{συν}(0) + i\,\text{ημ}(0)$

$k=1$: $z_1 = \text{συν}\left(\frac{2\pi}{\nu}\right) + i\,\text{ημ}\left(\frac{2\pi}{\nu}\right)$

Εκθετική Μορφή Μιγαδικών.

$z = \alpha + \beta i \qquad z = \rho(\text{συν}\theta + i\,\text{ημ}\theta)$

$|z| = \rho = \sqrt{\alpha^2 + \beta^2} \qquad \text{συν}\theta = \frac{\alpha}{\rho}$

$\text{ημ}\theta = \frac{\beta}{\rho}$.

$$z = |z|(\text{συν}\varphi + i\,\text{ημ}\varphi)$$

Εκθετική μορφή.

$$e^{i\varphi} = \text{συν}\varphi + i\,\text{ημ}\varphi$$

$z = |z| \cdot e^{i\varphi} \quad \leftarrow$ Εκθετική μορφή

$$|e^{i\varphi}| = |\text{συν}\varphi + i\,\text{ημ}\varphi| = \sqrt{\text{συν}^2\varphi + \text{ημ}^2\varphi} = \sqrt{1} = 1$$

- $e^{i0} = \text{συν}(0) + i\,\text{ημ}(0) = 1 + i \cdot 0 = 1$
- $e^{i\pi} = \text{συν}(\pi) + i\,\text{ημ}(\pi) = -1 + i0 = -1$
- $e^{i\frac{\pi}{2}} = \text{συν}\left(\frac{\pi}{2}\right) + i\,\text{ημ}\left(\frac{\pi}{2}\right) = 0 + 1i = i$
- $e^{-i\frac{\pi}{2}} = \left(e^{i\frac{3\pi}{2}}\right) = \text{συν}\left(-\frac{\pi}{2}\right) + i\,\text{ημ}\left(-\frac{\pi}{2}\right) = 0 + (-1) \cdot i = -i$

$$e^{-i\frac{\pi}{2}} = \left(e^{\frac{\pi i}{2}}\right) = \sigma\upsilon\nu\left(-\frac{\pi}{2}\right) + i\eta\mu\left(-\frac{\pi}{2}\right) = 0 + (-1)\cdot i = -i$$

$$e^{i\varphi} = \sigma\upsilon\nu\varphi + i\eta\mu\varphi$$

$$e^{-i\varphi} = e^{i(-\varphi)} = \sigma\upsilon\nu(-\varphi) + i\eta\mu(-\varphi)$$
$$= \sigma\upsilon\nu\varphi - i\eta\mu\varphi$$

$$e^{i\varphi} + e^{-i\varphi} = (\sigma\upsilon\nu\varphi + i\eta\mu\varphi) + (\sigma\upsilon\nu\varphi - i\eta\mu\varphi)$$
$$= 2\sigma\upsilon\nu\varphi$$

$$\sigma\upsilon\nu\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$e^{i\varphi} - e^{-i\varphi} = (\cancel{\sigma\upsilon\nu\varphi} + i\eta\mu\varphi) - (\cancel{\sigma\upsilon\nu\varphi} - i\eta\mu\varphi)$$
$$= 2i\eta\mu\varphi$$

$$\eta\mu\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

$$z_1 = |z_1|\cdot e^{i\varphi_1}, \quad z_2 = |z_2|\cdot e^{i\varphi_2}$$

$$z_1 \cdot z_2 = |z_1||z_2|\cdot e^{i\varphi_1}\cdot e^{i\varphi_2}$$
$$= |z_1||z_2|\, e^{i(\varphi_1+\varphi_2)}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\cdot e^{i(\varphi_1-\varphi_2)}$$

$$z^\nu = a \qquad a = \rho\,(\sigma\upsilon\nu\varphi + i\eta\mu\varphi) = \rho\cdot e^{i\varphi}$$

$$(|z|\cdot e^{i\theta})^\nu = \rho\cdot e^{i\varphi} \qquad e^{i\theta\nu} = e^{i\varphi}$$

$$|z| = \rho$$

$\varphi + 2k\pi$ ... $\varphi + 2k\pi$

$|z| = \rho$

$\nu\theta - \varphi = 2k\pi \Rightarrow \theta = \dfrac{\varphi + 2k\pi}{\nu}, \quad \theta_k = \dfrac{\varphi + 2k\pi}{\nu}$

$$z_k = \sqrt[\nu]{\rho} \cdot e^{\frac{\varphi + 2k\pi}{\nu} i}$$

Άσκηση

Να λυθεί η εξίσωση

$z^3 = 1 = a \qquad a = 1 = 1 + 0i \qquad |a| = 1$

$a = 1 \cdot e^{0}$

$z = |z| \cdot e^{i\theta}$

$z^3 = |z|^3 \cdot e^{i3\theta} = 1 \cdot e^{0} \Leftrightarrow |z|^3 = 1$

$|z| = \sqrt[3]{1}$

$\nu\theta - \varphi = 2k\pi \qquad \theta_k = \dfrac{2k\pi}{\nu}$

$\nu\theta - 0 = 2k\pi \qquad k = 0, 1, \ldots \qquad = \dfrac{2k\pi}{3}$

$\theta_0 = 1 \cdot e^{0} = 0 \Rightarrow z_0 = 1 \cdot e^{i0}$

$\theta_1 = \dfrac{2\pi}{3} \qquad z_1 = 1 \; e^{i\frac{2\pi}{3}}$

$\theta_2 = \dfrac{4\pi}{3} \qquad z_2 = 1 \; e^{i\frac{4\pi}{3}}$

Άσκηση

$-1 =$ τριγων.

$z^3 = -1 \qquad a = -1$

$a = -1 + 0i$ ($a_1 = -1$, $a_2 = 0$)

$|a| = \sqrt{a_1^2 + a_2^2} = \sqrt{(-1)^2 + 0^2} = 1$

$\left.\begin{array}{l} \sigma\upsilon\nu\varphi = \dfrac{a_1}{|a|} = \dfrac{-1}{1} = -1 \\ \ldots = \dfrac{a_2}{\ldots} = \dfrac{0}{\ldots} = 0 \end{array}\right\} \varphi = \pi$

$$\sigma v \nu \varphi = \frac{}{|a|} = \frac{}{1} \quad \Big\} \ \varphi = \pi$$

$$\eta \mu \varphi = \frac{a_2}{|a|} = \frac{0}{1} = 0$$

$$a = \sigma v \nu \pi + i \eta \mu \pi$$

$$a = 1 \cdot e^{i\pi} \quad (1)$$

$$z = |z| \cdot e^{i\theta} \qquad z^3 = |z|^3 \cdot e^{i3\theta} \quad (2)$$

$$z^3 = -1 \iff |z|^3 \cdot e^{i3\theta} = 1 \cdot e^{i\pi}$$

$$|z|^3 = 1 \iff |z| = \sqrt[3]{1} \qquad 3\theta - \pi = 2k\pi \iff$$

$$\theta_k = \frac{2k\pi + \pi}{3}$$

$$k = 0, 1, 2$$

$$z_k = \sqrt[3]{1} \cdot e^{i\frac{(2k+1)\pi}{3}} \qquad k = 0, 1, 2$$

$$z_0 = 1 \cdot e^{i\frac{\pi}{3}} = \sigma v \nu \frac{\pi}{3} + i \eta \mu \frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_1 = 1 \cdot e^{i\pi} = \sigma v \nu \pi + i \eta \mu \pi = -1 + i0$$

$$z_2 = 1 \cdot e^{i\frac{5\pi}{3}} = \sigma v \nu \frac{5\pi}{3} + i \eta \mu \frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$