Journal for Research in Mathematics Education
1985, Vol. 16, No. 1, 3–17

THE ROLE OF IMPLICIT MODELS IN SOLVING VERBAL PROBLEMS IN MULTIPLICATION AND DIVISION

EFRAIM FISCHBEIN, *Tel-Aviv University*
MARIA DERI, *University of Pisa*
MARIA SAINATI NELLO, *University of Pisa*
MARIA SCIOLIS MARINO, *University of Pisa*

Arithmetical operations were assumed to remain attached to primitive behavioral models that influence tacitly the choice of an operation even after the learner has had solid formal-algorithmic training. The model for multiplication was conjectured to be repeated addition, and two primitive models (partitive and quotative) were seen as linked to division. A total of 623 pupils enrolled in 13 Italian schools (Grades 5, 7, and 9) were asked to choose the operation needed to solve 26 multiplication and division word problems. Violations by the numerical data of the constraints imposed by the assumed tacit models (for instance, when the operator was a decimal number) constituted particular sources of difficulty at all three grade levels. The findings seemed to confirm the impact of the repeated addition model on multiplication and of the partitive model on division. The quotative division model influenced the pupils' choices only at the ninth-grade level.

The present study was inspired by previous findings on the difficulties children encounter when faced with verbal problems in multiplication and division. Bell, Swan, and Taylor (1981) have shown that when children are presented with a series of problems *with the same content,* they may change their minds about the operation needed to solve the problem, depending on the specific numerical data that are given. For instance, 12- to 15-year-old pupils were asked how to find the cost of 0.22 gallons of petrol if one gallon costs £1.20. (They were asked only to indicate the operation and not to perform the computation.) The most common answer was $1.20 \div 0.22$. When the same question was asked with "easy" numbers, such as £2 for the price of a gallon and 5 gallons for the amount of petrol, the pupils answered correctly: 2×5. When interviewed, the pupils did not consider it incongruous for the needed operation to change when the numbers changed. Bell et al. (p. 405) explained the mistaken response to the first problem as follows: The pupils correctly concluded that the cost of 0.22 gallons of petrol must be smaller than the cost of a gallon, and therefore they suggested division as the appropriate operation.

Hart (1980) reported a second finding related to the present study. She

This study was carried out in collaboration with the Didactical Group at the University of Pisa under the direction of Professor Giovanni Prodi. A previous version of the paper was presented at the Seventh International Conference for the Psychology of Mathematics Education, Shoresh Guest House, Israel, 24–29 July 1983.

found that 12- to 15-year-old pupils systematically avoided multiplying by fractions when solving a problem, even though that would be the simplest way to get a solution. They preferred more complicated strategies. For example, when given the problem "A 15-cm eel has 9 cm of food; how much food should be given to a 25-cm eel?", *no pupil multiplied 9 by 5/3*. Instead, the pupils used indirect strategies such as: 10 is two thirds of 15, two thirds of 9 is 6, and 25 is 15 + 10. Therefore, one has to add: 9 + 6 = 15 (Hart, 1981, p. 91).

In the eel problem, the answer must be a number larger than 9, and therefore according to Bell et al.'s (1981) interpretation, multiplication by 5/3 should have readily occurred to the pupils. In fact, that did not happen.

TACIT MODELS OF PROBLEM SITUATIONS

Consideration of such findings led us to the following hypothesis: *Each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model. Identification of the operation needed to solve a problem with two items of numerical data takes place not directly but as mediated by the model. The model imposes its own constraints on the search process.*

For example, suppose the concept of multiplication is intuitively attached to a repeated addition model so that 3 times 5 means 5 + 5 + 5. Under such an interpretation, the operator in multiplication can only be a whole number. A multiplication in which the operator is 0.22 or 5/3 has no intuitive meaning. If the numerical data of the problem do not fit the constraints of the model, the search process may not access the appropriate operation, and the solution effort may be diverted or simply blocked. The child either will resort to indirect ways to solve the problem—such as using an analogy or making a guess based on a global impression—or will simply not respond.

To say that multiplication by 0.22 or 5/3 has no intuitive meaning is not to say that it has no mathematical meaning. Children may know very well that 1.20×0.22 and $9 \times 5/3$ are legitimate mathematical expressions. But when given the petrol or eel problems cited above, *they may not be able to penetrate the problem to grasp the needed operation.* The way is blocked by the incongruity between the given numerical data and the specific constraints of the underlying tacit[1] model. The constraints may force the choice of an inadequate operation.

[1] The term *tacit* is used with approximately the same meaning as that of Polanyi (1969), who contends that cognitive decisions are never determined only by purely formal and explicit considerations but always depend on some unaccountable structures. We differ with Polanyi, however, in viewing tacit determinants of cognitive decisions as capable of becoming explicit and consequently under conscious control. Any didactical process requires such an assumption.

Factors Affecting Problem Difficulty

Researchers have described various factors that are presumed to be associated with the difficulties children have in solving word problems in arithmetic. The *familiarity of the context* and the *type of quantities involved* may affect problem difficulty, as may the *size and type of numbers used.* A problem may be more difficult if it contains whole numbers in the hundreds or above (Collis, 1975) or if it contains decimals (Bell et al., 1981). Another factor is the *relation between the situation referred to and the appropriate operation.* Hart (1980) found that multiplicative situations involving a Cartesian product were more difficult to interpret than situations reducible to repeated addition. Sometimes the action suggested by a problem conflicts with the operation actually required, as in a problem suggesting subtraction that is solved by addition or vice versa (Nesher, Greeno, & Riley, 1982; Nesher & Teubal, 1975). Children may have difficulties because of *certain effects rigidly associated with specific operations.* For example, they may believe that "multiplication makes bigger" and "division makes smaller" (Bell et al., 1981). Vergnaud (1983) has developed a theory of the epistemological obstacles that children encounter in learning multiplicative structures.

In addition to the factors that have received systematic study, tentative, ad hoc explanations of pupils' difficulties in solving arithmetic word problems have been offered from time to time. For instance: decimals are technically more difficult to handle than whole numbers; verbal cues may bias the solution process; a concrete context may facilitate finding a solution; pupils remain bound to the particular meaning originally attached to an operation; and most adolescents do not reach the stage of formal operations.

Our thesis is that the concept of an *intervening intuitive model* may explain in a coherent fashion most of the common difficulties children encounter when attempting to solve a problem requiring a single operation. The main exception is the difficulty caused by an unfamiliar text (terms used, situations referred to, etc.), which by itself may lead to considerable confusion and error.

Nature of the Tacit Models

We assume that the models attached to the arithmetical operations are basically behavioral in nature. That is, when trying to discover the intuitive model that a person tacitly associates with a certain operation, one has to consider some practical behavior that would be the enactive, effectively performable counterpart of the operation. This view is reminiscent of Piaget's theory, which says that every mental operation, including the operations of arithmetic, is developmentally rooted in practical situations. But in contrast to Piaget, we hypothesize that *the enactive prototype of an arithmetical* *operation may remain rigidly attached to the concept long after the concept has acquired a formal status.* In fact, this hypothesis may hold not only for arithmetical operations but for many other concepts as well.

These enactive models appear to act unconsciously to a great extent. They manipulate a person's problem-solving efforts "from behind the scene," and thus their impact can hardly be controlled by the solver. The models obey constraints imposed by their behavioral nature that do not always seem to fit the formal mathematical constraints of the corresponding operations.

MODELS ASSOCIATED WITH THE ARITHMETICAL OPERATIONS

Addition and Subtraction

The intuitive model associated with addition seems to be that of *putting together* two (or more) disjoint sets of objects to obtain a set that is their union. Subtraction, in contrast, appears to be given at least two behavioral interpretations: (a) *take away* (John has 10 marbles. He gives 4 to Jenny. How many marbles has he kept?), and (b) *building up* (John has 6 marbles. How many more does he need in order to have 10 marbles?). Much recent research has examined the implications of these models; for a recent account, see Carpenter, Moser, and Romberg (1982). The present investigation, however, was concerned with the less frequently studied operations of multiplication and division.

Multiplication

We hypothesized that the primitive model associated with multiplication is *repeated addition,* in which a number of collections of the same size are put together. Although other models, such as the rectangular pattern, have been proposed for teaching multiplication, their advocates recognize that the repeated addition model should be kept as a basic intuitive instructional device (Freudenthal, 1973, pp. 248–249). We do not claim that the teaching of multiplication must begin with this model or that characteristics of the child's mind presuppose it but only that for various reasons—which need to be investigated—this is the model that tacitly affects the meaning and use of multiplication, even in persons with considerable training in mathematics.

Under the repeated addition interpretation, 3×5 means $5 + 5 + 5$ or $3 + 3 + 3 + 3 + 3$. This interpretation does not view multiplication as commutative. One factor (the number of equivalent collections) is taken as the *operator;* the other (the magnitude of each collection), as the *operand.* The operand can be any positive quantity, but the operator must be a whole number. One cannot *intuitively* conceive of taking a quantity g 0.63 times or 3/7 times, whereas one can easily conceive of 3 times $0.63 = 0.63 + 0.63 + 0.63$, even if one cannot perform the operation. Because the operator is always a whole number, multiplication necessarily "makes bigger."

Division

Out of the several interpretations that might be given to division,[2] we hypothesized that two serve from childhood on to influence the evocation and use of the operation when a problem situation seems to call for division (Gibb, Jones, & Junge, 1959). The structure of the problem determines which model is activated.

Partitive division. In the first model, which might also be termed *sharing division,* an object or collection of objects is divided into a number of equal fragments or subcollections. The dividend must be larger than the divisor; the divisor (operator) must be a whole number; the quotient must be smaller than the dividend (operand).

Quotative division. In the second model, which might also be termed *measurement division,* one seeks to determine how many times a given quantity is contained in a larger quantity. In this case, the only constraint is that the dividend must be larger than the divisor. If the quotient is a whole number, the model can be seen as *repeated subtraction.*

METHOD

Subjects

The subjects were 628 pupils from 13 different schools in Pisa, Italy. There were 228 subjects in Grade 5 (10 or 11 years old), 202 in Grade 7 (12 or 13 years old), and 198 in Grade 9 (14 or 15 years old). In the Italian school curriculum, which all of these pupils followed, multiplying and dividing whole numbers is introduced in Grade 2, multiplying decimals in Grade 3, dividing decimals by whole numbers in Grade 4, and decimal divisors in Grade 5.

Instrument

A 42-item test was constructed containing 12 multiplication problems and 14 division problems. The remaining 16 items were problems in addition or subtraction that were included to reduce the likelihood of fortuitous correct answers. The data from the filler items are not reported here. An attempt was made to focus the pupils' attention on the numerical relationships by keeping the questions simple and direct. The problems referred to situations and quantities that should have been familiar to the pupils.

---

[2] Freudenthal (1973, pp. 252–254) criticizes the presentation of arithmetical division as related to only two practical interpretations: distribution (partitive) division and ratio (quotative) division. He notes that there are other types of division not reducible to these models and argues that emphasizing only the two may mislead the child. Although we agree with this view in principle, the other models he proposes—relating to velocity, area, volume, and so forth—are not suitable for the initial presentation of division to elementary school children. Such models require concepts and reasoning abilities that children do not possess at that age.

The items were mixed so that the addition and subtraction items were interspersed among the multiplication and division items. To reduce the number of items each pupil would receive and thereby reduce the effects of boredom and fatigue, the test was divided into two parts, Part A and Part B, containing 21 items each. To reduce any order effect, each part was ordered in two different ways, one the reverse of the other, which yielded four 21-item test forms.

The multiplication and division items are shown in Table 1. The numbering of the items is to facilitate the analysis and discussion; it does not correspond to the order on any of the test forms.

Procedure

The test was administered to the pupils in class groups. The forms were distributed at random in the classroom, and each pupil responded to just one form. The pupils were instructed to read the problem carefully before writing the answer. They were asked not to perform the actual calculation but only to indicate the operation used to solve the problem. A sample item was provided.

No order effect was found, so the data from the two forms for each part of the test were combined. The number of pupils responding to Part A was 116 at Grade 5, 94 at Grade 7, and 98 at Grade 9. The number responding to Part B was 112 at Grade 5, 108 at Grade 7, and 100 at Grade 9.

RESULTS

Multiplication

Table 2 shows the percentage distribution of responses to the multiplication items by grade and type of response. The most common errors are shown in the right-hand column, with the percentage in parentheses. The operand is always the first number and the operator the second regardless of the order in which the numbers appeared in the problem statement.

In the first two problems, both operator and operand are whole numbers. Almost all the pupils gave a correct response. The large operand in Problem 2 had no discernible effect on problem difficulty.

In Problems 3, 4, and 5, the numbers are the same, but in Problem 3 the operator is a whole number, and in Problems 4 and 5 it is a decimal. At each grade, Problems 4 and 5 were more difficult than Problem 3, which supports our view that a decimal operator is a source of difficulty. The extreme difficulty of Problem 5, especially for the younger pupils, may be attributable to their unfamiliarity with the content of the problem (particularly the notion of "detergent").

The difficulty of Problem 6 was similar to that of Problem 4, which again

Table 1

Problems Used in the Study

No.	Part	Statement[a]
		Multiplication problems
1	A	On the highway a car travels 2 km in 1 minute. If the speed of the car is constant, how far does it travel in 15 minutes?
2	B	1 kilo of oranges costs 1500 lire. What is the cost of 3 kilos?
3	A	From 1 quintal of wheat you get 0.75 quintal of flour. How much flour do you get from 15 quintals of wheat?
4	A	The volume of 1 quintal of gypsum is 15 cm^3. What is the volume of 0.75 quintal?
5	B	1 kilo of a detergent is used in making 15 kilos of soap. How much soap can be made from 0.75 kilo of detergent?
6	A	1 m of suit fabric costs 15 000 lire. How much does 0.75 m cost?
7	B	The price of 1 m of suit fabric is 15 000 lire. What is the price of 0.65 m?
8	A	For 1 cake you need 1.25 hg[b] of sugar. How much sugar do you need for 15 cakes?
9	B	For 1 kilo of cake you use 15 g of yeast. How much do you use for 1.25 kilos of cake?
10	B	1 piece of chocolate weighs 3.25 hg. How much do 15 pieces weigh?
11	A	A car goes 15 km on 1 L of fuel. How many km will it go on 3.25 L of fuel?
12	B	On 1 L of fuel a car goes 14 km. How many km will it go on 3.70 L of fuel?
		Division problems
13	A	With 75 pinks you can make 5 equal bouquets. How many pinks will be in each bouquet?
14	B	In 8 boxes there are 96 bottles of mineral water. How many bottles are in each box?
15	B	I spent 1500 lire for 3 hg of nuts. What is the price of 1hg?
16	A	15 friends together bought 5 kg of cookies. How much did each one get?
17	B	12 friends together bought 5 kg of cookies. How much did each one get?
18	A	To buy a dollar you need 1400 lire. How many dollars can you buy for 35 000 lire?
19	B	The walls of a bathroom are 280 cm high. How many rows of tile are needed to cover the walls if each row is 20 cm wide?
20	A	To wrap 5 equal packages requires 3.25 m of string. How much string is needed for each package?
21	A	5 friends together bought 0.75 kg of chocolate. How much does each one get?
22	B	5 bottles contain 1.25 L of beer. How much beer is in each bottle?
23	B	I spent 900 lire for 0.75 hg of cocoa. What is the price of 1 hg?
24	A	The walls of a bathroom are 3 m high. How many rows of tile are needed to cover the walls if the width of each row is 0.15 m?
	B	To trim 1 handkerchief you need 1.25 m of lace. How many handkerchiefs can you trim with 10 m of lace?
26	A	A tailor has 15 m of suit fabric. If 1 suit requires 3.25 m, how many suits can he make from the whole piece of fabric?

[a] The original problems were in Italian.
[b] 100 g ("etto" in Italian), a symbol familiar to Italian pupils.

Table 2
Distribution of Responses to Multiplication Problems

Problem no.	Operation	Grade	% correct	% no response	% incorrect	Most common errors (%)
1	2 × 15	5	84	5	10	15 ÷ 2 (6)
		7	96	3	1	
		9	98	0	2	
2	1500 × 3	5	97	1	2	
		7	91	1	8	1500 ÷ 3 (6)
		9	99	0	1	
3	0.75 × 15	5	79	7	14	15 ÷ 0.75 (5); 0.75 ÷ 15 (3)
		7	74	7	18	15 ÷ 0.75 (16)
		9	76	6	18	15 ÷ 0.75 (16)
4	15 × 0.75	5	57	9	34	15 ÷ 0.75 (16); 0.75 ÷ 15 (8)
		7	57	14	29	15 ÷ 0.75 (18); 0.75 ÷ 15 (9)
		9	46	14	40	15 ÷ 0.75 (28)
5	15 × 0.75	5	27	13	60	0.75 ÷ 15 (25); 15 ÷ 0.75 (20)
		7	18	28	55	15 ÷ 0.75 (24); 0.75 ÷ 15 (18)
		9	35	28	37	15 ÷ 0.75 (20)
6	15 000 × 0.75	5	53	7	40	15 000 ÷ 0.75 (28)
		7	57	5	37	15 000 ÷ 0.75 (31)
		9	52	6	42	15 000 ÷ 0.75 (37)
7	15 000 × 0.65	5	43	11	46	15 000 ÷ 0.65 (34)
		7	43	11	46	15 000 ÷ 0.65 (42)
		9	40	16	44	15 000 ÷ 0.65 (33)
8	1.25 × 15	5	84	4	11	1.25 ÷ 15 (7)
		7	91	2	6	1.25 ÷ 15 (4)
		9	94	1	5	1.25 ÷ 15 (3); 15 ÷ 1.25 (2)
9	15 × 1.25	5	54	18	29	1.25 ÷ 15 (15)
		7	38	25	37	1.25 ÷ 15 (19); 15 ÷ 1.25 (5)
		9	46	30	24	1.25 ÷ 15 (9); 15 ÷ 1.25 (5)
10	3.25 × 15	5	91	2	7	3.25 ÷ 15 (5)
		7	93	1	6	3.25 ÷ 15 (4)
		9	98	1	1	
11	15 × 3.25	5	80	4	16	3.25 ÷ 15 (9)
		7	85	1	14	3.25 ÷ 15 (7); 15 ÷ 3.25 (3)
		9	86	3	11	15 ÷ 3.25 (6); 3.25 ÷ 15 (5)
12	14 × 3.70	5	73	4	22	3.70 ÷ 14 (13)
		7	78	6	16	3.70 ÷ 14 (6); 14 ÷ 3.70 (4)
		9	87	6	7	14 ÷ 3.70 (3)

suggests that the size of the operand is relatively unimportant when the operand is a whole number.

We conjectured that a decimal operator like 0.65, being less familiar to pupils than 0.75, would cause them greater difficulty. That conjecture was borne out, as can be seen by comparing the results of Problems 6 and 7, two problems having exactly the same content and differing only in the decimal operator.

Problems 8 and 9 permitted another test of our view regarding decimal operators; the numbers are identical but with different roles. If only the "multiplication makes bigger" effect were operating, Problem 9 should not have been especially difficult, and if the presence of a decimal were the principal factor affecting difficulty, Problems 8 and 9 should have been of equal difficulty. The data, however, support our conjecture. Changing 1.25 from operand to operator cut the number of correct answers almost in half. Further, there was almost no progress with age.

We conjectured that when the whole part of a decimal is clearly larger than the fractional part, the pupil may treat it more like a whole number (as though the whole part "masks" or "absorbs" the fractional part). This conjecture, too, was confirmed. Problems 10 and 11 contain the same numbers in different roles, but this time the decimal is 3.25. The differences in difficulty between Problems 10 and 11 were smaller than those between Problems 8 and 9 or Problems 3 and 4. Although the decimal operator still appears as a source of difficulty, one can see that compared to decimals like 0.75, 0.65, or 1.25, a decimal like 3.25 has a slighter counterintuitive effect when playing the role of operator. A comparison of the results for Problems 9 and 11 suggests that this "absorption" effect may increase with grade, yielding an increasing frequency of correct responses.

In Problem 12, in contrast, the fractional part of the decimal is dominant—3.70. The data for Problems 11 and 12 suggest that, at least at Grades 5 and 7, the absorption effect did not operate as strongly in Problem 12 as in Problem 11. By Grade 9 the pupils had presumably learned to round decimals such as 3.70 to 4.

We conclude that the role of the decimal in the structure of a multiplication problem is clearly decisive in retrieving the correct operation. A multiplication problem becomes more difficult when the operator is a decimal (thus violating the repeated addition model). If the whole part of the decimal is large enough relative to the fractional part, however, an absorption effect appears to take place, which diminishes the counterintuitive effect of the decimal operator.

Division

Table 3 shows the percentage distribution of responses to the division items by grade and type of response. Again, the most common errors are shown in the right-hand column, with the percentage in parentheses.

Problems 13, 14, and 15 refer to partitive division situations and are in accordance with the presumed constraints of the corresponding division model. The problems were not difficult for the seventh or ninth graders. Only at Grade 5 did some differences appear. The presence of a large number (1500) or the symbol hg (for 100 g) in the statement of Problem 15 may have made it somewhat more difficult for the fifth graders.

Problems 16 and 17 violate the implicit model's rule that the dividend must

Table 3
Distribution of Responses to Division Problems

Problem no.	Operation	Grade	% correct	% no response	% incorrect	Most common errors (%)
13	75 ÷ 5 (partitive)	5	89	0	11	75 × 5 (10)
		7	93	0	7	75 × 5 (5)
		9	99	0	1	
14	96 ÷ 8 (partitive)	5	77	1	22	96 × 8 (19)
		7	90	0	10	96 × 8 (8)
		9	88	1	11	96 × 8 (10)
15	1500 ÷ 3 (partitive)	5	70	4	26	1500 × 3 (16)
		7	89	2	9	1500 × 3 (5)
		9	95	1	4	
16	5 ÷ 15 (partitive)	5	20	3	77	15 ÷ 5 (68)
		7	24	1	74	15 ÷ 5 (71)
		9	41	1	58	15 ÷ 5 (56)
17	5 ÷ 12 (partitive)	5	14	4	81	12 ÷ 5 (68)
		7	30	3	68	12 ÷ 5 (60)
		9	40	1	59	12 ÷ 5 (52)
18	35 000 ÷ 1400 (quotative)	5	68	3	28	35 000 × 1400 (18)
		7	79	6	15	35 000 − 1400 (6); 35 000 × 1400 (4)
		9	94	1	5	35 000 × 1400 (4)
19	280 ÷ 20 (quotative)	5	44	8	48	280 × 20 (41)
		7	77	7	16	280 × 20 (12)
		9	80	2	18	280 × 20 (14)
20	3.25 ÷ 5 (partitive)	5	73	4	22	3.25 × 5 (14)
		7	71	2	27	5 ÷ 3.25 (13); 3.25 × 5 (13)
		9	84	0	16	5 ÷ 3.25 (11)
21	0.75 ÷ 5 (partitive)	5	85	2	13	0.75 × 5 (7)
		7	77	3	20	5 ÷ 0.75 (17)
		9	83	1	16	5 ÷ 0.75 (11); 0.75 × 5 (5)
22	1.25 ÷ 5 (partitive)	5	66	2	32	1.25 × 5 (16); 5 ÷ 1.25 (16)
		7	74	1	25	5 ÷ 1.25 (17); 1.25 × 5 (8)
		9	70	1	29	5 ÷ 1.25 (23)
23	900 ÷ 0.75 (partitive)	5	22	22	55	900 × 0.75 (21)
		7	25	39	36	900 × 0.75 (7); 900 + 0.75 (5)
		9	40	29	31	900 × 0.75 (8)
24	3 ÷ 0.15 (quotative)	5	22	11	67	0.15 × 3 (53)
		7	38	9	53	0.15 × 3 (46)
		9	55	6	39	0.15 × 3 (37)
25	10 ÷ 1.25 (quotative)	5	31	4	65	1.25 × 10 (33); 1.25 ÷ 10 (19)
		7	63	4	32	1.25 ÷ 10 (14); 1.25 × 10 (11)
		9	79	1	20	1.25 × 10 (19)
26	15 ÷ 3.25 (quotative)	5	41	4	54	3.25 × 15 (32)
		7	62	7	31	3.25 ÷ 15 (14); 3.25 × 15 (14)
		9	90	1	9	3.25 × 15 (5); 3.25 ÷ 15 (3)

be larger than the divisor. This violation caused a drastic drop in the frequency of correct responses at all three grades, but the severity of the drop decreased as the grade increased. Most of the errors consisted of inverting the order of the terms, thus yielding an intuitive acceptable operation.

Problems 18 and 19 refer to quotative division, and they respect the constraints of the corresponding model. Nonetheless, they were more difficult than the partitive division problems with whole numbers (Problems 13 and 14), a finding that we had not predicted. Unfortunately, our test was not well constructed in regard to this point, and one cannot tell whether the drop was due to the increased size of the numbers or the shift in model.

In Problems 20, 21, and 22, the operand is a decimal and the operator is a whole number. In this respect, the three problems do not violate the rules of the partitive model. In contrast, they do violate the rule that the dividend should be larger than the divisor. Problems 16 and 17 also violate that rule, and the pupils' tendency was to reverse the roles of the numbers. Had they done that in Problems 20 to 22, however, they would have ended up with a decimal divisor! It appears that, faced with having to cope with a violation of the partitive model's rules, the pupils chose instead not to reverse the numbers. Perhaps surprisingly, then, their performance on Problems 20 to 22 was higher than on Problems 16 and 17, despite the fact that the divisor was greater than the dividend. It seems reasonable that the corrective mechanism by which a division like $0.75 \div 5$ (Problem 21) becomes intuitively feasible (in terms of the partitive model) consists in merely neglecting the decimal point (and thus seeing 0.75 as 75).

In Problem 23, the operator is a decimal, which violates a constraint of the partitive model. Comparing the data for Problem 23 with the corresponding data for Problems 13 and 14, which do not violate that constraint, one finds a drastic drop in the percentage of correct answers, a finding that supports our conjectures. The "operator not a decimal" rule of the partitive model seems to have a strong intuitive force.

We conjectured that in quotative division problems the operator might intuitively be a decimal, provided that the dividend is larger than the divisor. This conjecture was clearly confirmed for ninth graders only, as the data for Problems 24 to 26 indicate. The seventh graders appeared to be in a transition stage, and the fifth graders seemed dominated by the partitive model. That is, if the divisor was a decimal, it did not matter to these pupils whether the division was partitive or quotative.

In Problem 21, the decimal 0.75 was the operand, whereas in Problem 23 it was the operator. This shift in roles led to a sharp drop in the percentage of correct responses at all three grades.

The three quotative problems with decimal divisors (Problems 24 to 26) were much easier than the partitive problem with a decimal divisor (Problem 23), a finding that supports the assumption that in quotative problems the negative effect of the divisor being a decimal is diminished. The data for Problem 26, compared with those for Problems 24 and 25, suggest that, as was true for multiplication, the absorption effect (in which the whole part of the decimal masks the fractional part) operates for decimal divisors, but the effect is less clear for division.

The results for division lead us to conclude that our hypothesis of two basic intuitive models for division needs some modification. The hypothesis was confirmed at Grade 9 but not at Grade 5. We are led to conjecture that, initially, *there is only one intuitive primitive model for division problems*—the partitive model. With instruction, pupils acquire a second intuitive model—the quotative model. By Grade 9, the quotative model has become stable and influential, but pupils in Grade 7 are in a transitional stage with respect to the two models.

DISCUSSION

The basic assumption of the present research was that arithmetical operations are intuitively associated with some primitive behavioral models whose existence and influence the person may not be aware of. Such implicit models, acting to a great extent beyond any conscious formal control, may sometimes facilitate the course of problem solving, but very often they may slow down, divert, or even block the solution process when contradictions emerge between the model and the solution algorithm. Specifically, in the present investigation we assumed that (a) the primitive model of multiplication is repeated addition, and (b) there are two primitive models for division—the partitive and the quotative. We assumed that these models impose a number of constraints on the numbers used and their roles in the structure of the problem. If the data of a problem lead to a violation of one or more constraints, the solver may face difficulties in deciding on the appropriate arithmetical operation. Consequently, two problems may be operationally and even textually identical but may differ in difficulty as a function of the types of numbers used and their roles in the problem structure. In the repeated addition interpretation of multiplication, the operator must be a whole number and the product must be bigger than the operand. In the partitive interpretation of division, the divisor must be a whole number and both divisor and quotient must be smaller than the dividend. In the quotative interpretation of division, there is only one constraint—the divisor must be smaller than the dividend. In both multiplication and division, if the whole part of a decimal is significantly larger than the fractional part, the whole may intuitively "absorb" the fractional component, and thus the decimal acts psychologically as a whole number.

When one tries to represent explicitly what happens at the intuitive, tacit level in performing arithmetical operations, one sometimes gets a chain of mental transformations that are formally meaningless and algorithmically incorrect. When these transformations are interpreted within the constraints of the appropriate primitive model, however, the whole hypothetical chain becomes clear and consistent. Our findings support our belief that many of the difficulties children encounter when dealing with arithmetical concepts and operations can be explained in a similar fashion as arising from the conflict between formal algorithmic structures and related tacit, uncontrolled, primitive models. As just one example, consider Kieran's (1981) observation that children tend to confer on the equal sign not the formal properties of equality but rather those arising from its behavioral interpretation: By performing a certain activity (the left side), one gets a certain effect (the right side). Various implications follow from this interpretation, including, for instance, the assertion that a statement like $10 = 3 + 7$ has no meaning—a prediction that Kieran has confirmed with many children.

What are the sources of these primitive models? Two explanations seem plausible. The most direct explanation is that a model reflects the way in which the corresponding concept or operation was initially taught in school. As the first interpretation learned by the child, it tends to become strongly rooted in his or her mental behavior. In Bachelard's (1980) phrase, it is "the experience that someone gets before and above any critical attitude" (p. 23). A second explanation is that these primitive models are so resistant to change and so influential because they correspond to features of human mental behavior that are primary, natural, and basic. People naturally tend to interpret facts and ideas in terms of structured models that are behaviorally and enactively meaningful. This tendency may maintain the presence of the primitive models above and beyond any formal rules one has learned. In our view, both explanations are correct.

The teacher's choice of the repeated addition model for multiplication and the partitive, and then the quotative, model for division is made for both epistemological and ontogenetic reasons. Such models have consistently been chosen as initial didactical devices because they correspond best to the mental requirements of elementary school children at the concrete operational period and because they provide the most natural way of understanding the new concept. Piaget was right when he described "operations" as derived by the internalization of external actions.

If our interpretation is correct, teachers of arithmetic face a fundamental didactical dilemma. On the one hand, if one continues to introduce the operations of multiplication and division through the models described above, one will create—as our findings demonstrate—strong, resistant, and, at the same time, incomplete models that soon will come to conflict with the formal concepts of multiplication and division. On the other hand, if one tries to avoid building the ideas related to arithmetical operations on a foundation that is behaviorally and intuitively meaningful, one certainly will violate the most elementary principles of psychology and didactics. This is one instance of a general dilemma facing mathematics teachers. They know the problem. But they have assumed, in line with Piagetian theory, that with the emergence of the period of formal operations children would naturally become less dependent on intuitive justifications and limitations and more open to formal considerations. The primitive models used in initial instruction would then lose much of their impact on the adolescent's mathematical reasoning.

Our findings show that the dilemma is much more profound than it might appear at first glance. *The initial didactical models seem to become so deeply rooted in the learner's mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct.* The development of formal operational reasoning in adolescence does not by itself resolve the dilemma. Adolescents and adults alike continue to face difficulties when they have to solve elementary problems in arithmetic with numerical data that lead to conflict between the correct operation and the constraints of the corresponding tacit model.[3] A next step in this line of research would be to attempt to provide learners with efficient mental strategies that would enable then to control the impact of these primitive models.

REFERENCES

Bachelard, G. (1980). *La formation de l'esprit scientifique.* Paris: Librairie Philosophique J. Vrin.

Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational Studies in Mathematics, 12,* 399–420.

Carpenter, T. P., Moser, J. M., & Romberg, T. A. (Eds.). (1982). *Addition and subtraction: A cognitive perspective.* Hillsdale, NJ: Erlbaum.

Collis, K. F. (1975). *Concrete and formal operations in school mathematics.* Melbourne: Australian Council for Educational Research.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, Holland: Reidel.

Gibb, E. G., Jones, P. S., & Junge, C. W. (1959). Number and operation. In P. S. Jones (Ed.), *The growth of mathematical ideas, Grades K–12* (24th Yearbook, pp. 7–64). Washington, DC: National Council of Teachers of Mathematics.

Hart, K. M. (1980). *Secondary school children's understanding of mathematics* (Report of the Concepts in Secondary Mathematics and Science Programme). London: University of London, Centre for Science Education.

Hart, K. M. (Ed.). (1981). *Children's understanding of mathematics: 11–16.* London: John Murray.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12,* 317–326.

[3]Two allegations in the present paper are not based on systematic evidence: (a) that the tacit models assumed by us continue to affect the solution process even in adults, and (b) that these models act in a rather unconscious way. We support our claims with the following observations. We have had many opportunities to ask university students, teachers, and researchers to solve some of the typically "difficult" problems that appear in our tests. Most of these adults were not able to indicate *directly* the operation that would solve the problem. They had to resort to analogies and proportion strategies, and they could report the *indirect* procedure they had used. Furthermore, *all* of these adults were surprised to learn that the difficulty derives from the impact of a certain intuitive model. They certainly were not aware of the still-active role that these old, primitive interpretations played in their thinking.

Nesher, P., Greeno, J. G., & Riley, M. S. (1982). The development of semantic categories for addition and subtraction. *Educational Studies in Mathematics, 13,* 373–394.

Nesher, P., & Teubal, E. (1975). Verbal cues on an interfering factor in verbal problem solving. *Educational Studies in Mathematics, 6,* 41–51.

Polanyi, M. (1969). *Knowing and being.* London: Routledge & Kegan Paul.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). London: Academic Press.

[Received November 1983; revised July 1984]

AUTHORS

EFRAIM FISCHBEIN, Professor, School of Education, Tel-Aviv University, Tel-Aviv, 69978, Israel

MARIA DERI, Teacher and Researcher, Didactical Seminary, University of Pisa, Piazza die Cavalieri 2, Pisa I 56100, Italy

MARIA SAINATI NELLO, Teacher and Researcher, Didactical Seminary, University of Pisa, Piazza die Cavalieri 2, Pisa I 56100, Italy

MARIA SCIOLIS MARINO, Teacher and Researcher, Didactical Seminary, University of Pisa, Piazza die Cavalieri 2, Pisa I 56100, Italy