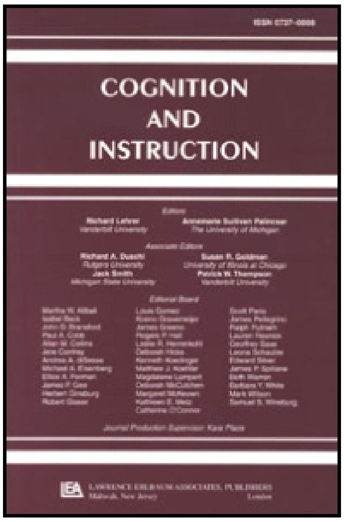

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How Many *Decimals* Are There Between Two *Fractions*? Aspects of Secondary School Students' Understanding of Rational Numbers and Their Notation

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We present an empirical study that investigated seventh-, ninth-, and eleventh-grade students' understanding of the infinity of numbers in an interval. The participants ($n = 549$) were asked how many (i.e., a finite or infinite number of numbers) and what type of numbers (i.e., decimals, fractions, or any type) lie between two rational numbers. The results showed that the idea of discreteness (i.e., that fractions and decimals had "successors" like natural numbers) was robust in all age groups; that students tended to believe that the intermediate numbers must be of the same type as the interval endpoints (i.e., only decimals between decimals and fractions between fractions); and that the type of interval endpoints (natural numbers, decimals, or fractions) influenced students' judgments of the number of intermediate numbers in those intervals. We interpret these findings within the framework theory approach to conceptual change.

The purpose of this study was to investigate the development of secondary school students' understanding of the dense ordering of rational numbers, as reflected in their judgments of the number of numbers in an interval. Rational (and also real) numbers are densely ordered (hereafter, dense[1]) in the sense that between any two numbers there is always an intermediate number, which implies that there are infinitely many intermediates. On the contrary, natural numbers are discrete, in the sense that for any natural number, there is a unique successor. Therefore, between any two natural numbers there is a finite number (possibly zero) of numbers. Prior research has shown that students mistakenly assign this property of discreteness to rational and real numbers as well, not only at elementary and secondary education (Hartnett & Gelman, 1998; Malara, 2001; Merenluoto & Lehtinen, 2002; Neumann, 2001; Pehkonen, Hannula, Maijala, & Soro, 2006; Vamvakoussi & Vosniadou, 2004, 2007), but at the tertiary education level as well (Giannakoulias, Souyoul, &

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[1]We stress that the terms "discrete" and "dense" are used here with respect to the usual order relation. In this sense, natural numbers are discrete, whereas rational and real numbers are dense, the latter being not merely dense, but continuous (e.g., Lucas, 2000). This clarification is in order because the term "discreteness" is sometimes associated with the countability of the rational numbers (i.e., the fact that one-to-one correspondence can be established between natural and rational numbers). However, if rational numbers are "re-ordered" to match the natural numbers, then two successive numbers in this respect are not successive in terms of the usual ordering.

Zachariades, 2007; Tirosh, Fischbein, Graeber, & Wilson, 1999). Typically this phenomenon is explained as an adverse effect of students' prior knowledge and experience with natural numbers.

In the present article we argue that the idea of discreteness is only one of the factors that must be taken into consideration in order to explain students' difficulties with the density of rational numbers. Another such factor is the problem of interpreting rational number notation. Furthermore, we argue that these two problems are related to a more general problem of conceptual change in the transition from natural to rational numbers.

Let us start with an example: Panos, an articulate, self-confident ninth grader, very competent in mathematics, is answering a set of questions regarding the number of numbers between two rational numbers (Vamvakoussi & Vosniadou, 2004). "Between 0.001 and 0.01," he says, "there are nine numbers." Then he goes on to the next question and without hesitation, he states that between 3/5 and 4/5 there is an infinite number of numbers. Following a prompt by the interviewer, Panos explains: "Between 0.001 and 0.01 there are nine numbers. Or maybe ten—I'm not so sure about that. But if you convert them to fractions, you can find more numbers in between. You can find infinitely many numbers."

Panos seems to believe that decimals preserve, at least locally, the discrete ordering of natural numbers, but that fractions do not—despite the fact that he clearly knows that the particular decimals can be converted to fractions. To the mathematically versed person, for whom decimals and fractions are interchangeable representations of the same mathematical objects (i.e., rational numbers), this student's response contains an obvious inconsistency. To Panos however it does not, presumably because from his point of view it is not necessary for fractions and decimals to behave in the same way, at least in the particular context.

To the best of our knowledge, there are no studies investigating systematically the effect of the symbolic representation of numbers on students' judgments about the number of numbers in an interval, although some dispersed findings from relative studies indicate that there might be such an effect. For example, Tirosh et al. (1999) noted that the prospective teachers that participated in their study responded better in the case of decimals, as compared to fractions: 40% answered that there are infinitely many numbers between 0.23 and 0.24, whereas only 24% gave the same response in the case of 1/5 and 1/4. The reverse was reported by Giannakoulias et al. (2007): 87% of the participants, who were first year university students in a mathematics department, answered correctly in the case of 1/3 and 2/3, as compared to 71% in the case of 0.1 and 0.11. Vamvakoussi and Vosniadou (2007) found that many ninth and eleventh graders changed their answers from "a finite number" to "infinitely many intermediate numbers" depending on the type of the interval endpoints (decimals or fractions). Unlike Panos, the "finiteness" answer appeared more often in the case of fractions, than in the case of decimals. Furthermore, even students who consistently gave an "infinitely many" response were reluctant to accept that the intermediate numbers could be of any symbolic representation. A similar finding is reported by Neumann (2001), who noticed that seventh graders had difficulties accepting that there could be a fraction between 0.3 and 0.6.

To sum up, prior research has shown that the density property of rational numbers is difficult for students to understand, with the idea of discreteness as a major stumbling block. However, as the example of Panos and the relative evidence presented earlier shows, in order to understand better how students cope with this difficult notion, it is necessary to take into consideration the effect of the symbolic representation of rational numbers on their judgments, something not investigated systematically so far.

In the section that follows we present a short mathematical review of rational numbers, focusing on the particular characteristics that differentiate them from natural numbers. We then discuss some students' difficulties in the transition from natural to rational numbers. We acknowledge that we are citing only a subset of the vast work on rational numbers.

WHAT IS A RATIONAL NUMBER?

From a (school) mathematics point of view, a rational number is usually defined as a number that is or can be expressed in the form *a*/*b*, where a and b are integers, and b is non-zero. Because integers, terminating decimals, and also repeating decimals can be expressed in the form *a*/*b*, they are rational numbers.[2] One could say that 0.5, 500/1.000 and also 1/2 or 7/14, are *all* rational numbers. However, a closer examination shows that they all have the same value, thus the accurate thing to say would be that these are alternative representations of the *same* rational number. A mathematician would define this rational number as the equivalence class of pairs $[a,b]$ such that $2a = b$, and would say that all the above are representatives of this class. The set of rational numbers contains the natural numbers as a subset and the non-natural rational numbers can be ordered, added, and multiplied in ways that are consistent with the natural numbers. Thus in the context of advanced mathematics, rational numbers are conceptualized as abstract individual entities, which take their meaning within a formal system (Dörfler, 1995). This kind of abstraction allows for conceptualizing rational numbers as unique objects, invariant under different symbolic representations and uniting natural and non-natural numbers as members of the same family.

This perspective on rational number is typically not available to students who encounter rational numbers in the form of fractions and decimals in school contexts.[3] Although also called "numbers," these new constructs are in many ways different than natural numbers. Indeed, natural numbers are associated with absolute quantity, and they are typically answers to a "how many" question. They are discrete, in the sense that each has a unique successor. The unit is the building block of numbers by means of its repetition. Within the natural number set, addition and multiplication always "make bigger," whereas subtraction and division always "make smaller"; and each number is associated with one symbol. On the other hand, rational numbers represent quantitative relations and they answer a "how much" rather than a "how many" question. They are dense; none has a unique successor. With rational numbers, the unit is infinitely divisible and new numbers are generated by means of division. Addition and multiplication do not necessarily "make bigger" and subtraction and division do not necessarily "make smaller." And as already discussed, they can be written in different symbolic representations.

The expansion of the natural numbers set to include non-natural numbers as well entails substantial differences in the meaning of the term number and consequently in the behavior of

[2]In the following, when we use the term "decimal" we will refer only to decimals that are elements of the rational numbers set, as opposed to decimals that are not rational because they do not repeat, such as π, or .0101101110 . . . .

[3]For the purposes of this article, we use the following the term "fraction" to refer to numbers that are actually presented symbolically in the form *a*/*b*, and the term "rational number" to refer to any number that is or can be expressed in the form *a*/*b*, regardless of its symbolic representation. Thus, the term "rational numbers" is not used in the strict mathematical sense of the word.

numbers. It is precisely in these differences that one can trace many of the pervasive difficulties that students experience with rational numbers (Smith, 1995).

STUDENTS' DIFFICULTIES WITH RATIONAL NUMBERS

Rational numbers are notoriously difficult for students to master. Natural number knowledge interference (i.e., applying natural number knowledge in situations when it is not appropriate) and problems with rational number notation are widely recognized as two major sources of students' difficulty with rational numbers (Moss, 2005; Ni & Zhou, 2005; Smith, Solomon, & Carey, 2005). Moreover, these two problems seem to be connected, since there is a good deal of evidence showing that students' prior knowledge and experience with natural numbers has an adverse effect on the interpretation of rational number notation. Thus students are found, for example, to interpret the symbol *a*/*b* as two unrelated natural numbers, or focus on the additive, rather than the multiplicative, aspect of the relation between *a* and *b*; or they appear to think that "longer decimals are bigger" or "the bigger the terms, the bigger the fraction" (Moskal & Magone, 2000; Lamon, 1999; Moss, 2005; Stafylidou & Vosniadou, 2004).

Yet another source of difficulty with rational number notation is the multiplicity of symbolic representations of rational numbers. This relates to a more general issue of representations and their relation to the objects of study which is of particular importance in mathematics, since it is highly related to the ontology of mathematical objects. As Duval (2006) argues, access to mathematical objects is available only through signs and semiotic representations—one can never observe the number 1/2 in the environment—but mathematical objects should not be confused with the representations used. This presents students with a major conceptual difficulty, especially when it comes to realizing that different representation may have the same referent (Markovits & Sowder, 1991).

Mathematics educators and researchers have noticed that students find it difficult to conceptualize rational numbers as a unified system of numbers (Kilpatrick, Swafford, & Findell, 2001) and that they may treat decimals and fractions as if they were different kinds of numbers, rather than interchangeable representations of the same numbers (e.g. Khoury & Zazkis, 1994; O'Connor, 2001). Markovits and Sowder (1991) found that the majority of the middle-grades participants of their study were not able to deal effectively with a series of tasks targeting their understanding of the relation between decimals and fractions. For example, students were reluctant to order a series of numbers presented in decimal and fractional form and they either ordered them separately, or they explicitly stated that this could not be done. Confusing the number and its representations has also been noticed in the case of equivalent fractions: Elementary and secondary students are reported to state that equivalent fractions, like 1/2 and 2/4, count as different numbers (Mitchell, 2005; Vamvakoussi & Vosniadou, 2004). One could argue that students are, in this respect, similar to young children who, according to Sfard (2007), use the word "number" the same way mathematically versed persons use the word "numeral."

As we have discussed, the broadening of numbers to include non-natural numbers essentially changes what counts as a number and how it behaves. Prior research has shown that one of the main reasons for students' difficulties with rational numbers is that they rely on natural number reasoning in situations when this is no longer appropriate and this includes their interpretation of rational number notation (e.g., Moss, 2005; Ni & Zhou, 2005). Such problems may reflect deep conceptual difficulties with the rational number concept, rather than mere confusion with symbols or occasional intrusions of natural number knowledge (Ni & Zhou, 2005; Smith et al., 2005). Furthermore, we believe that students' difficulties with the multiplicity of rational number representations are another facet of the complex interplay between students' prior knowledge about natural numbers and their understanding of rational number notation.

We claim that we can make progress toward a theory of rational number understanding by analyzing students' difficulties in the context of the framework theory approach to conceptual change (Vosniadou, 2007; Vosniadou, Vamvakoussi, & Skopeliti, 2008). In the following we elaborate the key assumptions and predictions of this framework. We ground the discussion on the case of the development of the number concept, drawing on relevant empirical evidence from the rational number learning and teaching literature. We argue that we can provide an account of how Panos, the student presented earlier, has developed his particular understanding of rational numbers. Furthermore, findings of the empirical study presented in this article will demonstrate that this is not merely an idiosyncratic understanding of rational numbers.

THE FRAMEWORK THEORY APPROACH TO CONCEPTUAL CHANGE

The framework theory approach to conceptual change (Vosniadou, 2007; Vosniadou et al., 2008) was originally developed to explain students' persistent misconceptions in science learning, but has been recently used productively in the area of mathematics learning as well (Vosniadou & Verschaffel, 2004; Vosniadou, 2007). A key assumption is that from early on children organize their interpretations of common everyday experiences in the context of lay culture into few, relatively coherent, categories or *framework theories*. Thus, with respect to the different perspectives on conceptual change, our theoretical position emphasizes *coherence*, rather than *fragmentation* [for a thorough discussion, see diSessa (2006, 2008) and Vosniadou et al. (2008)]. The term *theory* is not used here to denote an explicit, well formed, and socially shared construct. Rather, this term is used to indicate a relatively coherent and principle-based system, which is generative in that it allows children to make predictions and explanations and deal with unfamiliar problems. Evidence coming from cognitive developmental research (see Carey, 1991; Carey & Spelke, 1994; Keil, 1994) suggests that there are at least four framework theories in the domains of physics, psychology, language, and, finally, number, on which we focus in the following.

Infants and young children have strong dispositions to learn rapidly and readily about number (Carey & Gelman, 1991; Carey, 1985; Gelman, 1990). Although some researchers argue that early quantitative representation is limited to discrete quantity, and thus the natural number concept is privileged, this issue is still controversial (see Ni & Zhou, 2005, for a thorough discussion). It appears that some pre-instructed ideas pertaining to rational number also exist in children, even pre-schoolers. Moss and Case (1999), for example, refer to two *primitive psychological units*, one for proportional evaluation, and one for splitting (see also Smith et al., 2005). However, the externalization and systematization of such intuitions is typically not socially supported in the first years of a child's life (Greer, 2004). On the contrary, the development of the natural number concept is mediated from early on by cultural representational tools, language being probably the most prominent (Carey, 2004), and also practices such as finger counting (Andres, Di Luca, & Pesenti, 2008).

Based on relevant empirical studies, Gelman (1994, 2000) concludes that children by four or five years of age possess a principled understanding of number as counting number, which supports them to "invent solutions to novel counting problems, detect errors in counting trails generated by others and make up counting algorithms to solve simple addition and subtraction problems, at least for a limited range of the numbers" (Gelman, 1994, p. 68). The counting algorithm also supports children to build the successor principle, which in turn may help them to infer that there are infinitely many natural numbers (Hartnett & Gelman, 1998). In other words, initial understandings of numbers support children to reason about natural numbers, learn about their properties, and build strategies in relation to natural number operations. This continues during the first years of instruction, when, in the context of natural number arithmetic, addition and subtraction are conceptualized in terms of counting [see, for example, Resnick (1989) for a report of counting-based strategies], multiplication is viewed as repeated addition (e.g. Fischbein, Deri, Nello, & Marino, 1985), and numbers can be ordered by counting-based strategies.

Thus it seems that, before they are exposed to rational number instruction, students have formed a rather coherent explanatory framework of number as *counting number* which, in terms of the theoretical framework that we propose, constitutes an initial, domain-specific *theory* of number (Vamvakoussi & Vosniadou, 2004, 2007; Vosniadou et al., 2008). Arguably, many components of children's initial framework theory of number are experientially based, building on the multiplicity of perceptual and sensory experiences obtained via observation and interaction with physical objects—an assumption consistent with the "knowledge in pieces" perspective (diSessa, 1993, 2008). However, we assume that these are integrated from early on into larger conceptual structures under the influence of lay culture and early instruction which, as already discussed, support the externalization, systematization, and validation of children's experiences.

Students' initial framework theory of number encompasses several background assumptions and beliefs that underlie their expectations about what counts as a number and how it is supposed to behave. As we suggested earlier, we assume that within students' initial framework theory, numbers are essentially natural numbers, which obey the successor principle and behave accordingly with respect to ordering, operations, and so on. When non-natural numbers are introduced in the curriculum in the form of decimals and fractions, although they have the label *numbers* attached to them, they are clearly not numbers in the same sense as students' initial framework theory of number. How do students make sense of these new entities?

The framework theory approach to conceptual change is a constructivist approach. We believe that students typically employ additive mechanisms of learning (e.g., assimilation, accommodation, internalization) to enrich gradually their initial framework theory of number with new information about non-natural numbers. When the incoming information, however, is in contrast with what is already known, the use of additive mechanisms can destroy the coherence of the original knowledge structure and the learning process may result in fragmentation, internal inconsistency, and the formation of misconceptions. Thus, although students' exposure to symbolic and verbal knowledge systems as presented via instruction and texts usually triggers top-down coherence in their knowledge organization (diSessa, 1993), we believe that fragmentation and inconsistency can be a first product of instruction, in cases when a rather coherent, but incompatible, structure is already established.

On the other hand, similar to Smith, diSessa, and Roschelle (1993), we do not expect students to hold unitary, isolated, and context-independent misconceptions. Thus, students may be found to use decimals and fractions adequately, depending on the affordances of the specific contexts (i.e., familiarity, type of representations employed, etc.). Nevertheless, success in one or more specific contexts should not lead us to discount a failure in a different context, because it may actually point to a persistent conceptual difficulty (Sophian, 1997).

Thus within the framework theory approach to conceptual change, the phenomenon of students' misconceptions due to faulty natural number reasoning (Moss, 2005; Ni & Zhou, 2005) can be explained not as occasional intrusions of students' prior knowledge, but as an indication that students draw heavily on their initial understandings of number to make sense of rational numbers. This assumption is further corroborated by the fact that students are found to deal successfully with tasks that are congruent with their natural-number-based expectations (e.g. Nunes & Bryant, 2008). It has also been shown that when new information about rational numbers is presented in a context that allows for counting-based reasoning, it is more accessible to students, one such example being the part–whole aspect of fraction (Moss, 2005; Mamede, Nunes, & Bryant, 2005).

A particular strength of our theoretical framework, as compared to other approaches to conceptual change, is that it provides an account of the transition process from an initial to a more sophisticated understanding of counterintuitive concepts, predicting the generation of *synthetic conceptions*. Synthetic conceptions of the rational numbers set combine natural numbers, decimals, and fractions and their properties into aggregate views that are far from the abstract view of rational numbers as a unified, densely ordered system of numbers and may only be locally consistent. In the context of ordering, local consistency means that judgments about order are consistent within specific types of endpoints, (i.e., natural numbers, decimals, or fractions) but inconsistent across these types. We assume that moving from a natural number to a rational number perspective would be a gradual and time consuming process. Synthetic conceptions represent an intermediate state of knowledge that creates a bridge between the student's initial perspective of number and the intended scientific perspective, which is not yet available to the student. The prediction that students create synthetic conceptions tests the assumption of the theory that students rely mainly on enrichment type, additive mechanisms to add new information to prior knowledge. Finally, the identification of synthetic conceptions can potentially prove useful to inform instruction about the intermediate states of understanding (see Vosniadou et al., 2008 for a detailed discussion).

We illustrate the notion of synthetic conceptions by revisiting the case of Panos, the student presented earlier. This student has expanded the use of the term number to include non-natural numbers. He has also learned a lot about these new numbers: He knows enough to sustain his belief that there are infinitely many numbers between fractions; he knows that decimals can be converted to fractions; he is also able to find some more numbers between the two given decimals, most probably by adding one more decimal digit. Nevertheless, he still thinks that decimals preserve, at least locally the discrete ordering of rational numbers, but fractions do not. Moreover, he is not bothered by this inconsistency, presumably because from his point of view it is not necessary for fractions and decimals to behave in the same way, at least with respect to ordering. In this sense, Panos sees decimals and fractions as two different kinds of numbers, rather than interchangeable representations of the same numbers. Thus, an important aspect of the rational number concept is not yet available to Panos. Nevertheless, his synthetic conception of decimals and fractions as two different subsets of rational numbers provides an intermediate state of illusionary coherence that conceals the conflict between inconsistent beliefs and allows for the student to deal with the tasks at hand and move on in the knowledge acquisition process.

Thus we argue that for Panos, expanding the initial framework theory of numbers to include non-natural numbers by gradually "adding" relative new information, produced the synthetic conception of decimals and fractions as different, unrelated "subsets" of rational numbers. We believe that this synthetic conception is not an idiosyncratic construction of this particular student, but one of a set of possible similar conceptions that have this "additive" quality and which are likely to appear in other students, as well. For instance, we expect that students will also have great difficulty conceptualizing natural and non-natural numbers as members of the same family (see also Sfard, 1991). There is a common source of difficulty behind the conceptualization of decimals and fractions as interchangeable representations of the same numbers, and also of natural and non-natural numbers as members of the same category. This is because the transition from students' initial theory of number to the rational number perspective requires, as already discussed, an essentially new and far more abstract conceptualization of number which is substantially different than students' initial theory of number.

For learners to accept that natural and non-natural numbers are members of the same category, natural numbers need to be re-conceptualized from a rational number perspective, which is not yet available to students. Moreover, this re-conceptualization entails changes in the behavior of natural numbers that are in sharp contrast with students' initial framework theory of numbers. An additional difficulty is that students cannot *abandon* their initial (natural) number perspective for the more sophisticated (rational) number perspective. Although subsumed under the broader category of the rational numbers, natural numbers *within the natural numbers set* preserve all their particular characteristics. Thus, rather than abandoning their natural number reasoning, students are required to distinguish the contexts and the conditions that call for natural or rational number reasoning. The ability to move flexibly among different perspectives is a characteristic of experts that takes a great deal of time and personal effort to develop, but also extensive cultural support [see, for example, Merenluoto and Palonen (2007) for an account of the complex interplay between the development of real number understanding and the initiation in the mathematical community].

On the other hand, conceptualizing decimals and fractions as interchangeable representations of the same numbers requires moving beyond their differences in form and behavior and focusing on their common reference (Duval, 2006; Markovits & Sowder, 1991), that is, realizing that they refer to the same *quantitative relation*, or at a more abstract level, the same *rational number*. Again, the rational number perspective is not yet available to students. Because the conceptual background necessary to sustain the realization that decimals and fractions refer to the same mathematical objects is not in place, the multiplicity of representations becomes an additional burden for the students (see also Ni & Zhou, 2005).

To summarize, we have argued that the process of acquiring new information about rational numbers, under the constraints of the initial framework theory of number, may produce a class of synthetic conceptions of the rational numbers as consisting of different, unrelated, subsets of numbers, along the natural/non natural, and also the decimal/fraction distinction. We expect that contexts that call for a conceptualization of rational numbers as a unified system of numbers are very likely to reveal such synthetic conceptions on the part of students.

The notion of density is interesting in this respect first because it violates a fundamental presupposition of students' initial framework theory of number, namely, the idea of discreteness (see also Ni & Zhou, 2005; Smith et al., 2005). Second, it requires a conceptualization of rational numbers as a unified system of numbers, and not as consisting of different, unrelated sets of numbers.

The empirical study presented in this article is the third in a series of studies in which we investigated secondary students' understanding of the number of numbers in an interval (Vamvakoussi & Vosniadou, 2004, 2007). Our main hypothesis has been that students would often answer that there is a finite number of intermediate numbers and, in particular, that there is no other number between two pseudo-successive numbers like .005 and .006, or 1/3 and 2/3. In the first study we conducted individual interviews with ninth graders (approximate age 15 years). In the second study we administered open-ended and multiple-choice questionnaires to ninth and eleventh graders.

These two studies showed that the idea of discreteness was robust and also gave strong indications that the symbolic representation of the numbers involved affected students' judgments about the number of intermediate numbers, as well as about their type. However, the latter issue required more systematic investigation. This is because the number of items we used was small and not counterbalanced with respect to decimal and fraction items. Another concern was that the open questionnaires in our second study elicited a considerable number of indeterminate answers. More specifically, many students answered that there are "more" or "many" intermediate numbers, but did not specify how many and/or of what type, although they were specifically instructed to. Finally, neither in our first, nor in our second study had we included items with pairs of natural numbers. Thus, although we had some information regarding the decimal/fraction distinction, we had none with respect to the natural/non- natural distinction.

THE PRESENT STUDY

In the present study we aimed, first, to replicate the results of our previous studies showing that students frequently answer that there is a finite number of intermediate numbers in an interval. We hypothesized that age and level of instruction would have a positive effect on students' performance, but that the constraint of discreteness would remain strong even for the older students.

Second, we aimed to show that understanding about the infinity of numbers in any given interval is not an "all or nothing situation"; rather, there are intermediate levels of understanding, where the presupposition of discreteness is lifted for some, but not all types of numbers. We hypothesized that there would be a mathematically naïve level of understanding, where students would consistently answer that there is a finite number of numbers in any given interval, and a sophisticated level, where students would accept that there are infinitely many numbers, regardless of their symbolic representation. In between, we expected to find a number of students whose responses would be internally inconsistent, and also students whose responses would reflect synthetic conceptions indicating some local coherence where rational numbers would be systematically treated as a set of different, unrelated sets of numbers, along the natural/non-natural, and also the fraction/decimal distinction. Specifically, we predicted that:

a) Students would be more willing to accept the presence and the variability in terms of symbolic representation of intermediate numbers between natural, than between non-natural numbers. This prediction was based on the fact that whole numbers are typically used as points of reference when non-natural numbers are presented (e.g., on the number line) and also because the view of non-naturals as numbers inserted between whole numbers is compatible with students' experiences with measurement and tools like the ruler.

b) There would be students who would give consistently different answers for intervals defined by decimals and fractions, in terms of their ordering (dense or discrete).

c) Students would answer frequently that there are only decimals between decimals and only fractions between fractions.

Finally, we tested for contextual effects (i.e., whether slight variations of the presentation of tasks might have an effect on students' responses). We were interested in students' responses in the case of two pseudo-successive numbers (such as 2.1 and 2.2), as compared to the case of two numbers that do not appear to be successive (such as 2.1 and 2.5). An obvious, yet trivial, effect might be that students would be more likely to answer that there are *some* intermediate numbers in the latter case. However, there is another possible effect that is more interesting. We hypothesized that the pseudo-successive numbers would trigger more "infinitely many intermediates" answers, than the non pseudo-successive ones. This is because the salient sequence of intermediate numbers in the latter case might trick students, who otherwise might have given an "infinitely many intermediates" response, to answer that there is a finite number of intermediate numbers. This kind of contextual inconsistency would further establish the robustness of the idea of discreteness.

In comparison to our previous studies (Vamvakoussi & Vosniadou, 2004, 2007), we broadened our age sample to include seventh graders in addition to ninth and eleventh graders. We considered seventh graders an interesting group to study and compare with older students, because, based on their elementary school curriculum, they had had access to all the procedural knowledge necessary to deal with density-related tasks. They had been exposed to extensive instruction about decimals and fractions (ordering, operations, turning decimals into fractions and vice versa, etc.), but unlike ninth graders, they had not been introduced to the term rational numbers and had not been exposed to instruction about real numbers. We note that eleventh graders were the only age group who had been explicitly taught about the interrelations between the various subsets of real numbers through Venn diagrams.

Based on our experience with the open-ended questionnaires of the 2007 study, we chose to use only multiple-choice items in order to avoid indeterminate data. In addition, based on prior research we expected that multiple-choice items including the correct alternative would be easier for students (Vosniadou, Skopeliti, & Ikospentaki, 2004). Therefore, they would provide a stronger test of our hypotheses. Besides including natural number items, the number of the fraction and decimal items was increased in order to investigate the consistency of students' responses within and across different types of numbers.

METHOD

Participants

The participants of this study were 181 seventh graders (mean age 12 years 6 months), 166 ninth graders (mean age 14 years 7 months), and 202 eleventh graders (mean age 16 years 5 months).

About half of the participants were female. They came from six middle class public schools in the Athens area.

Materials

We designed two multiple-choice questionnaires (hereafter Q_1 and Q_2), each consisting of 14 questions. All questions were of the same type, asking how many numbers there are in a given interval (see Table 1).

As shown in Table 1, both questionnaires consisted of 4 item groups, differentiated by the type of the interval endpoints, namely two natural numbers (Nn, 2 items), a natural number and a decimal (NnDec, 4 items), two decimals (Dec, 4 items), and two fractions (Fra, 4 items). The Nn group was the same in Q_1 and Q_2. Two of the items of the remaining groups (NnDec, Dec, Fra,) were common in Q_1 and Q_2. The given intervals in these common items were defined by pseudo-successive numbers (such as 0.005 and 0.006, or 1/3 and 2/3). The remaining items were also defined by pseudo-successive numbers in Q_1, but not in Q_2. Thus, Q_1 consisted only of items involving pseudo-successive numbers, whereas in Q_2 half of the items (with the exception of Nn items) involved numbers that were not pseudo-successive.

Table 1 also presents the types of multiple choices offered to the students which were the same for all 14 items. Every choice (with the exception of A_1 and A_7) required students to decide two issues: (1) the number of numbers in the interval and (2) the type of numbers in the interval. With respect to (1), there were the following alternatives: (i) there is no other number in the interval

TABLE 1
Design of the Two Types of Questionnaire (Q_1, Q_2)

Item Groups	*Type of the Interval Endpoints*	*Items*: *Q_1 and Q_2: Common Items*	*Q_1 vs. Q_2: Different Items* Q_1	Q_2
Natural Numbers (Nn)	2 Natural numbers	0–1 99–100		
Natural number-Decimal (NnDec)	Natural-Decimal with 1 decimal digit	0.9–1	6–6.1	6–6.4
	Natural-Decimal with 3 decimal digits	0.009–1	7–7.001	7–7.003
Decimals (Dec)	2 Decimals with 1 decimal digit	0.1–0.2	2.4–2.5	2.4–2.7
	2 Decimals with 3 decimal digits	0.005–0.006	3.123–3.124	3.123–3.126
Fractions (Fra)	Two same denominator fractions	1/3–2/3	3/5–4/5	1/5–4/5
	Two different denominator fractions	1/7–1/6	1/8–1/7	1/9–1/7

Multiple choices

A_1	There is no other number (Fin_0, *Decimals* or *Fractions*)
A_2	There is a finite number of decimals ($Fin_{\neq 0}$, *Decimals*)
A_3	There is a finite number of fractions ($Fin_{\neq 0}$, *Fractions*)
A_4	There are infinitely many decimals (Inf-, *Decimals*)
A_5	There are infinitely many fractions (Inf-, *Fractions*)
A_6	There are infinitely many numbers and they can have various forms (e.g. fractions, decimals, decimals with infinitely many decimal digits). (Inf, *All types of numbers*)
A_7	I don't agree with any of the above. I believe that ... ...

(Fin_0), (ii) there is a finite number of numbers in the interval ($Fin_{\neq 0}$), (iii) there are infinitely many numbers of a specific type (Inf-), and (iv) there are infinitely many numbers of all types (Inf). With respect to (2), there were the following alternatives: (i) decimals, (ii) fractions, and (iii) all types of numbers.

As we have discussed, we expected one effect of the type of questionnaire would be that there would be fewer Fin_0 answers in Q_2 than Q_1. However, the possible effect that was of interest to us was that students would provide Inf- or Inf responses more frequently in Q_1 than in Q_2, because they would be tricked by the presence of a salient sequence of numbers on the non pseudo-successive items in Q_2.

Procedure

The questionnaires were administered during regular school hours in the classrooms. Half of the students in each class received Q_1 and the other half received Q_2. As a result, the participants were almost equally divided between the Q_1 and Q_2 conditions (375 and 371 students, respectively).

One of the researchers read aloud some clarifications/instructions, which were also written on the first page of the questionnaire. These were the following: (a) "The term 'numbers' refers to real numbers—all numbers that you know of are real numbers." (b) "We say that a number is between two other numbers, if it is greater than the first and smaller than the second." (c) "The particular multiple choice responses used in the questionnaires were provided by students of your age. Do not hesitate to express a different opinion, because it might be better or more accurate than the given answers." And (d) "You can choose only one of the answers." In case they found that more than one of the given answers fit, students were instructed to use the free space (A_7) to write it and explain why. Finally, the expression "a finite number of numbers" was explained to our participants as a specific amount of numbers, all of which could be written down one by one. We note that in the case of seventh graders, the term *real numbers* was not used; instead, the instruction was phrased as follows: "The term 'numbers' refers to all numbers that you know of."

The students had 45 minutes available to complete the questionnaires.

RESULTS

Effects of Type of Questionnaire and Grade on Performance

To test for the effect of the type of questionnaire, we first examined the percent of the Fin_0 responses in the total of questions combined. As expected, the Fin_0 response was more frequent in Q_1 (15.22%), than in Q_2 (10.44%). To test for other possible effects of the questionnaire type, we controlled for this difference via merging the response-types Fin_0 and $Fin_{\neq 0}$ into one, hereafter FIN response type. By doing so, the response *there is no other number* between, for instance, 6 and 6.1 (Q_1) was comparable to the response *there is a finite number of numbers* in the corresponding item in Q_2 (e.g., between 6 and 6.4). Students' mean performance in the 14 questions was computed by scoring the FIN answer as 1, the Inf- answer as 2, and the Inf answer as 3.

Table 2 presents the mean and standard deviation of students' mean performance, by grade and type of questionnaire.

TABLE 2
Mean and Standard Deviation of Students' Mean Performance, by Grade and Type of Questionnaire

Grade	*Questionnaire*	*Mean*	*Std. Deviation*	*n*
7th	Q_1	1.621	.398	91
	Q_2	1.578	.440	90
	Total	1.599	.418	181
9th	Q_1	1.923	.701	84
	Q_2	1.975	.703	82
	Total	1.948	.701	166
11th	Q_1	2.030	.664	101
	Q_2	1.896	.670	101
	Total	1.963	.669	202
Total	Q_1	1.863	.626	276
	Q_2	1.815	.636	273
	Total	1.839	.631	549

The total mean scores were subjected to a 3 (grade) $\times$ 2 (questionnaire type) analysis of variance. The results showed main effects only for grade [$F(2, 548) = 20.923$, $p < .001$] (see also Table 3).

A Tukey HSD test showed that the ninth and eleventh graders performed significantly better than the seventh graders (at the .001 level) under both conditions (Q_1/Q_2).

Because the questionnaire type effect on the total performance was not found to be significant, this condition was not taken into consideration in the analyses that follow.

The Presupposition of Discreteness

Table 4 presents the frequency and percent of each response type (FIN, Inf-, Inf), for all 14 questions combined, for all students, by grade.

As can be seen in Table 4, the presupposition of discreteness reflected in the FIN response was particularly strong in seventh grade (53.8% of the answers given in all questions combined), and remained strong in ninth (42.25%), as well as in eleventh grade (39.99%).

TABLE 3
ANOVA Results Table: 3 (grade) $\times$ 2 (questionnaire type) Analysis of Variance on Students' Total Mean Performance

Source	*Sum of Squares*	*df*	*Mean Square*	*F*	*Sig.*	η^2
Grade	15.511	2	7.755	20.923	.000	.072
Questionnaire	.238	1	.238	.642	.423	.001
Grade*Questionnaire	.794	2	.397	1.071	.343	.004
Error	201.263	543	.371			
Total	2074.128	549				
Corrected Total	217.861	548				

TABLE 4
Frequency and Percent of each Response Type in the Total of Answers (14 × number of students) for all Questions (14 items) Combined, per Grade

Response type	*7th grade* (n = *2534)*	*9th grade* (n = *2324)*	*11th grade* (n = *2828)*
FIN	1364	982	1131
	53.83%	42.25%	39.99%
Inf-	668	399	574
	26.36%	17.17%	20.30%
Inf	451	916	1091
	17.80%	39.41%	38.58%
No answer	51	27	32
	2.01%	1.16%	1.13%

Effects of the Type of the Interval Endpoints

Effects on Performance. Students' mean performance in the four items groups (Nn, NnDec, Dec, & Fra$_{)}$ was subjected to a repeated measures analysis of variance with item group type as within subjects and grade as between subjects factor. The results showed main effects for item group type [$F(3, 546) = 52.447$, $p < .001$], and for grade [$F(2, 546) = 25.426$, $p < .001$].

Figure 1 presents the estimated marginal means of students' performance in the different item groups, per grade.

For all age groups, mean performance was higher in Nn and gradually decreased in the cases of NnDec and Dec. The lowest performance was found in Fra. Pair-wise comparisons showed

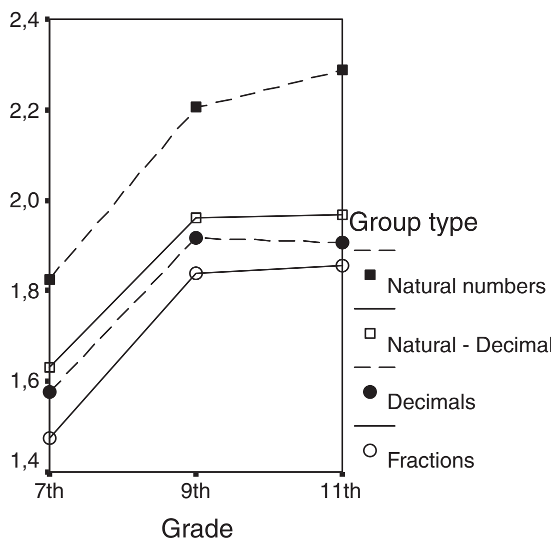

FIGURE 1 Estimated marginal means of students' mean performance in the different item groups, per grade.

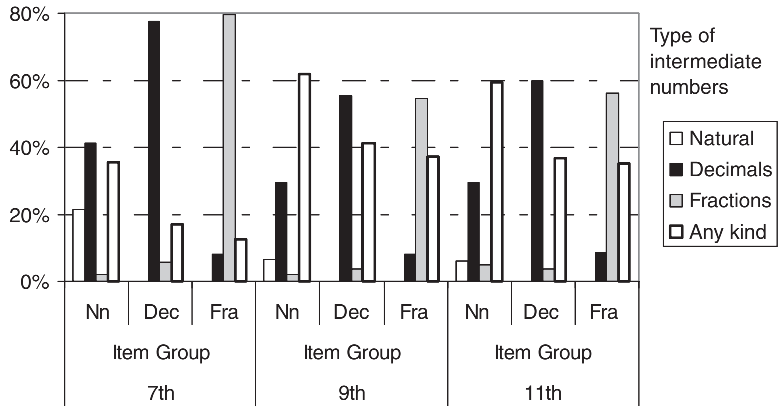

FIGURE 2 Percent of each response type, per item group, by grade (null answers excluded).

that these differences were significant (at the .001 level) in the case of Nn, as compared to all other item groups, and that seventh graders performed significantly better (at the .01 level) in NnDec and Dec than in Fra. Finally, ninth and eleventh graders performed significantly better (at the .001 level) than seventh graders in all item groups.

Effects on the Type of the Intermediate Numbers. We then examined the effect of the interval endpoints on students' responses about the type of the intermediate numbers. Figure 2 presents the percent of each response type, per item group, by grade.

As Figure 2 shows, students tended to choose intermediate numbers according to the symbolic representation of the interval endpoints. On the other hand, all age groups were more likely to accept that between two natural numbers there can be any type of numbers, regardless of their representation. In addition, it should be noted that a response different than "any type of number" between natural numbers was typically "decimals" and only very rarely "fractions."

It can be noticed that all age groups showed a very similar pattern of responses, in the sense that (a) all three age groups gave more sophisticated answers in Nn, as compared to all other item groups and (b) all three age groups gave less sophisticated answers in Fra, as compared to all other item groups.

Cross-Sectional Developmental Results

We categorized the participants based on the total number of FIN (i.e., finite number of numbers) and INF (i.e., infinitely many numbers, namely Inf- and Inf answers combined) responses they gave in Nn, Dec, and Fra (a total of 10 items). The NnDec items were not included in this analysis, because we were mainly interested in the consistency of students' responses for the intervals defined by numbers of the same type.

We placed students who gave at least 7 FIN answers in the 10 questions in Level I and students who gave at least 7 INF answers in Level III. The remaining students were placed at Level II.

TABLE 5
Frequency and Percent of Students in Levels I, II, and III

	Grade			
Level	*7th*	*9th*	*11th*	*Total*
Level I	76	49	60	181
	42%	29.5%	29.7%	33%
Level II	68	42	50	160
	37.6%	25.3%	24.8%	29.1%
Level III	37	75	92	204
	20.4%	45.2%	45.5%	37.2%
Total	181	166	202	549

This is a rough categorization, aiming at distinguishing between students on the "finite" and the "infinite" side and those who were in between. We chose a 70% criterion because we wanted Level I to include students who gave mostly FIN responses, but also those who gave an INF answer to the natural number items but not to the remaining ones. Level III was defined with a 70% criterion for INF answers. Table 5 presents the frequency and percent of students at Level I, II, and III.

As can be seen in Table 5, Level I consisted mostly of seventh graders (42%), while Level III consisted mostly of ninth and eleventh graders (45.2% and 45.5%, respectively). Nevertheless, about one third of both the ninth and eleventh graders were still at Level I. In other words, a considerable number of the older students gave a FIN answer in at least 7 out of the 10 items in question. This finding strengthens the hypothesis that the presupposition of discreteness remains strong even for older students.

Response Types Within Level

Figures 3, 4, and 5 present the percent of the FIN, Inf-, Inf responses within each item group (Natural numbers, Decimals, Fractions) at Levels I, II, and III, respectively.

As expected, the FIN answer was dominant at Level I, for all item groups, the lowest percent appearing for natural numbers. At the intermediate Level II, the FIN answer remained dominant for decimals and fractions, but not for natural numbers. In the latter case, the dominant answer was the Inf answer. At Level III, the Inf answer was dominant for all item groups, the highest percent appearing again for natural numbers. As far as decimals and fractions are concerned, with the exception of Level I, where the percent of FIN answers was slightly lower for fractions, more sophisticated answers were given in the case of decimals. This is particularly evident at Level II.

Consistency of Responses Within and Across Types of Numbers

We then examined the consistency of students' responses within each item group, in terms of the finite/infinite number of numbers distinction. We defined students' profiles based on their

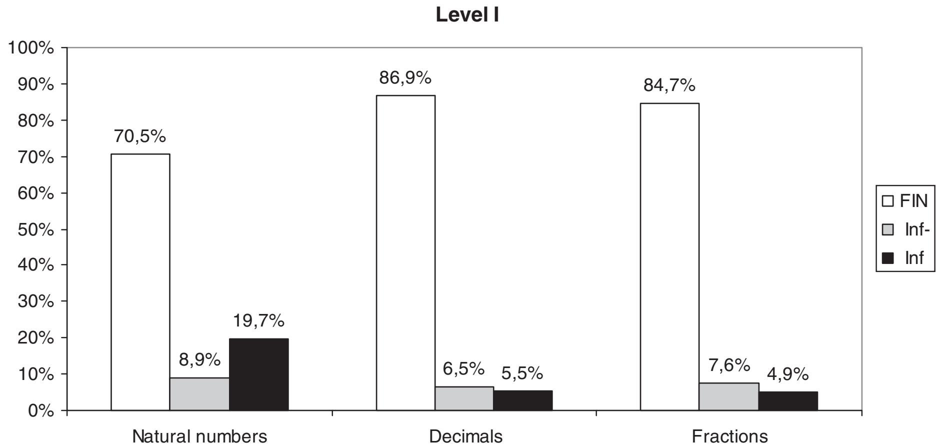

FIGURE 3 Percent of the FIN, Inf-, and Inf responses within each item group (Natural numbers, Decimals, Fractions), at Level I.

answers in all 10 questions. We considered that a student gave a consistent answer for decimals or fractions if she gave FIN (or INF) answers in at least three out of four items. As far as natural numbers are concerned, we considered that a student gave a consistent answer if she answered both relative items with FIN (or INF) answers. We will refer to this kind of consistent responses as FIN* (or INF*, respectively). We will refer to the remaining as mixed (F/I) responses. Table 6 presents students' responses in terms of consistency/inconsistency within each item group (natural numbers, decimals, and fractions).

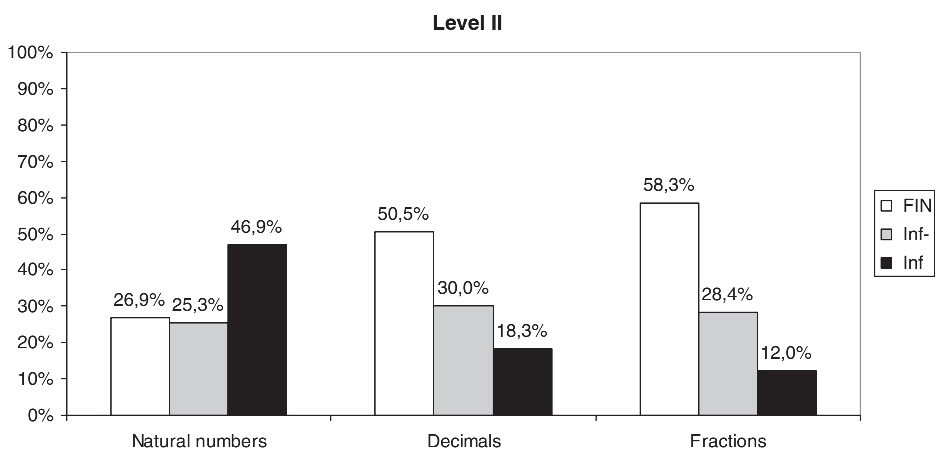

FIGURE 4 Percent of the FIN, Inf-, Inf responses within each item group (Natural numbers, Decimals, Fractions), at Level II.

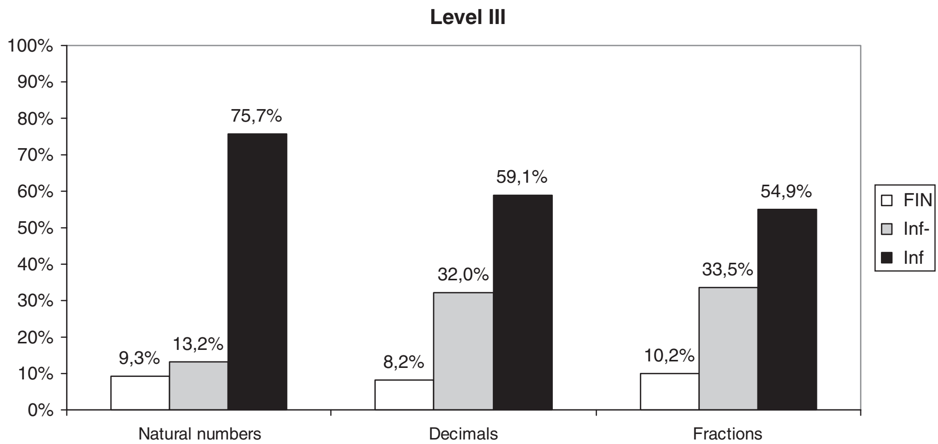

FIGURE 5 Percent of the FIN, Inf-, Inf responses within each item group (Natural numbers, Decimals, Fractions), at Level III.

We note that students answered quite consistently within each item group, since the percent of inconsistent responses was about 24% for natural numbers, 14% for decimals and 16% for fractions. However, natural and non-natural numbers were clearly treated in a different way since less FIN* and more INF* answers were provided in the first case. In fact, we found that 40% of the INF* answers for natural numbers came from Level I and Level II students, as compared to 22.58% and 19.09% for decimals and fractions, respectively. On the other hand, decimals and fractions did not differ considerably in this respect, despite the fact that the FIN* response appeared more frequently for fractions, whereas the INF* appeared more frequently for decimals (see Table 6).

However, this does not necessarily imply that more sophisticated reasoning about decimals and fractions develops in parallel for the individual student: From the data presented in Table 6 for decimals, the percent of consistent responses (either FIN* or INF*) was in total about 86%. Similarly, for fractions, the percent of consistent responses was in total about 84%. However, the corresponding percent dropped to 59%, when considering the consistent responses for both decimals and fractions combined.

TABLE 6
Students' Responses in Terms of Consistency/Inconsistency within Each Item Group

Item Group	*FIN**	*INF**	*F/I*	
Natural numbers	131	288	130	549
	23.86%	52.46%	23.68%	100.00%
Decimals	224	248	77	549
	40.80%	45.17%	14.03%	100.00%
Fractions	239	220	90	549
	43.53%	40.07%	16.39%	100.00%

TABLE 7
Categorization of the Students, Based on the Consistency of Their Responses (FIN*, INF*) Across the Three Item Groups (Natural numbers, Decimals, Fractions)

	Grade			
Category	*7th*	*9th*	*11th*	*Total*
*All FIN** (Initial)	28	28	23	79
	15.47%	16.87%	11.39%	14.39%
FIN for Decimals & Fractions*	33	13	26	72
	18.23%	7.83%	12.87%	13.11%
FIN for Decimals, INF* for Fractions*	10	4	13	27
	5.52%	2.41%	6.44%	4.92%
INF for Decimals, FIN* for Fractions*	27	9	14	50
	14.92%	5.42%	6.93%	9.11%
INF for Decimals & Fractions*	19	11	4	34
	10.50%	6.63%	1.98%	6.19%
*All INF**	16	52	72	140
	8.84%	31.33%	35.64%	25.50%
Remaining	48	49	50	147
	26.52%	29.52%	24.75%	26.78%
Total	181	166	202	549

Student Profiles

In Table 7 we present a finer categorization of the students, based on the consistency of their responses across the three item groups (natural numbers, decimals, and fractions).

Students in the *All FIN** category gave a FIN* response for natural, as well as non-natural numbers. These students appear constrained by the presupposition of discreteness across all types of numbers.

Students in the *FIN* for Decimals and Fractions* category appeared to be still constrained by the presupposition of discreteness for non-natural numbers, but gave more sophisticated answers in the case of natural numbers. Interestingly, the reverse is observed in the *INF* for Decimals and Fractions* category. Students in these two categories combined (19.30% in the total of students) treated natural numbers differently than non-natural numbers. On the other hand, students in the *INF* for Decimals, FIN* for Fractions* category seem to think that decimals are dense, whereas fractions are discrete. The reverse holds for students in the *FIN* for Decimals, INF* for Fractions* category. In sum, 14.03% of the students responded differently for decimals, as compared to fractions. These findings provide support for the hypothesis that some students tend to think differently for the intervals in terms of their structure (discrete/dense), depending on the interval endpoints. Usually, this difference is in favor of natural numbers and decimals.

Students in the *All INF** category gave an INF* response for natural, as well as non-natural numbers, indicating that they have overcome the idea of discreteness. However, a considerable number of these students were still constrained by the belief that the interval endpoints and the intermediate numbers should be of the same symbolic representation. Table 8 presents a categorization of the *All INF** students, based on the consistency of their responses across the three item groups.

TABLE 8
Categorization of the All INF* Students, Based on the Consistency of Their Responses (Inf-*, Inf*) across the Three Item Groups

	Grade				
Category	*7th*	*9th*	*11th*	*Total*	*Overall Total (N = 549)*
Inf- for Decimals & Fractions*	6	6	14	26	26
	37.50%	11.54%	19.44%	18.6%	4.74%
Inf- for Decimals, Inf* for Fractions*	2	1	1	4	4
	12.50%	1.92%	1.39%	2.86%	0.73%
Inf for Decimals, Inf-* for Fractions*	0	2	2	4	4
	0%	3.85%	2.78%	2.86%	0.73%
All Inf (Sophisticated)*	5	35	39	79	79
	31.25%	67.31%	54.17%	56.43%	14.39%
Mixture of Inf, Inf-	3	8	16	27	27
	18.80%	15.38	22.22%	19.29%	4.92%
Total (*All INF**)	16	52	72	140	140
					25.50%

As can been seen in Table 8, only students in the *Sophisticated* category (56.43% within the *All INF** category, 14.39% in the total of 549 students) answered consistently that there are infinitely many numbers, of all types, for both natural and non-natural number endpoints. The remaining *All INF** category students generally gave a mixture of Inf and Inf- responses.

We note that the pattern of more sophisticated answers for natural, than for non-natural numbers appeared again with 18.6% of the students in this category (4.74% in the total of students) responding Inf-* for decimals and fractions, (*Inf-* for Decimals and Fractions* category). In addition, 5.72% of the students in the category (1.46% in the total of students) answered differently for decimals, as compared to fractions. These students answered that there can be infinitely many numbers, of all types, between decimals, but infinitely many fractions between fractions (or vice versa).

We further note that, as expected, a smaller percent of seventh, than ninth and eleventh, graders participated in the *All INF** category in general, and in the *Sophisticated* category in particular. On the other hand, although a higher percent of eleventh graders than ninth graders participated in the *All INF** category, there are relatively more ninth graders in the *Sophisticated* category (35/166 = 21.10%, as compared to 39/202 = 19.31%).

Finally, the students who were not placed in any of the previous categories gave F/I responses within at least one item group. Fifty of them (9.12% in the total of students) gave INF* responses in the case of natural numbers, whereas another 61 (11.11% in the total of students) gave FIN* answers for decimals or fractions

DISCUSSION

Summary of the Results

In this study we investigated students' understanding of the density of rational numbers, through their judgments about the number of numbers in different intervals. In line with prior research, the results showed that the idea of discreteness is a major constraint on students' understandings of density (Giannakoulias, Souyoul, & Zachariades, 2007; Hartnett & Gelman, 1998; Malara, 2001; Merenluoto & Lehtinen, 2002; Neumann, 2001; Pehkonen et al., 2006; Tirosh et al., 1999; Vamvakoussi & Vosniadou, 2004, 2007). In addition, the results showed that there is indeed an effect of the type of the interval endpoints (natural numbers, decimals, or fractions) on students' judgments on the number, as well as the type of intermediate numbers. More specifically, students tended to believe that the intermediate numbers must be of the same type as the interval endpoints (i.e., only decimals between decimals and fractions between fractions), and did not necessarily treat natural numbers, decimals and fractions in the same way with respect to the number of intermediate numbers. These findings strengthen and extend the results of our previous studies (Vamvakoussi & Vosniadou, 2004, 2007) and are compatible with evidence indicating that students find the nature of the relation between decimals and fractions difficult to grasp (Khoury & Zazkis, 1994; Markovits & Sowder, 1991; Mitchell, 2005; Neumann, 2001; O'Connor, 2001). They also agree with the observation made by Kilpatrick et al. (2001) that it is difficult for students to see the rational numbers set as a unified system of numbers.

In a more general fashion, these findings relate to the problem of natural number knowledge interference in rational number tasks, and also the problem of interpretation of rational number notation, which prior research identifies as two major difficulties in the learning of rational number (e.g., Moss, 2005). In this article we have argued that these problems are interconnected and reveal deep conceptual difficulties with the rational number concept, rather than occasional intrusions of prior knowledge or mere confusion with symbols (see also Ni & Zhou, 2005; Smith et al., 2005). The results of this study showed that both the idea of discreteness and the interpretation of rational number notation need to be considered in order to account for students' understandings of the infinity of numbers in an interval. This aspect of students' understanding of the density property of rational number has not been systematically investigated before, thus we believe that this study adds to the relative knowledge.

In what follows we will draw on our findings to provide an account of how this understanding develops, from the perspective of the framework theory approach to conceptual change. This analysis will give some insight into the possible intermediate states of students' understandings of the relations between natural numbers, decimals, and fractions via the notion of synthetic conceptions, which is particular to our theoretical framework, as compared to other approaches to conceptual change (Vosniadou et al., 2008). We acknowledge that these data come from a cross-sectional study. This poses certain limitations on the attempts to describe the developmental progression, which we also discuss below. Nevertheless, we believe that they provide a plausible path of development, to be validated by further research.

Interpretation of the Results Within the Framework Theory Approach to Conceptual Change

As we have discussed, the framework theory approach to conceptual change assumes that students, before they are exposed to instruction about rational number, have consolidated a complex system of interrelated beliefs and background assumptions about what counts as a number and how it behaves, namely an initial framework theory of number within which number is conceptualized as *counting number*. Decimals and fractions are in many respects different from natural numbers, yet there are also presented as numbers; moreover, there are differences between decimals and fractions with respect to notation, ordering, and arithmetic operations, even if only at the procedural level. A mathematically educated person, even if not a trained mathematician, can think of and talk about rational numbers as non-tangible individual entities, invariant under different representations, focusing on the commonalities, rather than the differences of natural and non-natural numbers, such as that they both correspond to points on the number line and that they relate through the same operations (Dörfler, 1995). The conceptualization of rational numbers as a unified system, involving the ability to differentiate number from its representations (Duval, 2006), we argue, requires fundamental ontological and epistemological changes to take place in students' conceptual organization of number (i.e., requires conceptual change).

The framework theory approach to conceptual change is a constructivist approach that assumes that students will build on their prior knowledge to make sense of rational numbers. Because new information about rational number is, however, incompatible with students' understandings of number, we predicted that the acquisition process: (a) will be slow and gradual, (b) may at first result in fragmentation and inconsistency, and (c) is likely to produce synthetic conceptions.

The results of this study supported the predictions of the framework theory approach to conceptual change, in the context of students' understanding about the infinity of numbers in an interval. First, they suggested that the process of development is a slow and gradual process and not an all-or-nothing situation; and, second, that it proceeds from a rather coherent view of number as discrete, to fragmentation, and then to coherence again, and not from fragmentation to coherence (e.g., diSessa, 1993). As shown in Figures 3, 4, and 5, students at Level I, who are mostly the younger students, have a rather coherent view of number as discrete. Seventy-nine of these 181 students (see Table 7) gave FIN* responses within and across all types of numbers (natural numbers, decimals, and fractions). This response pattern becomes fragmented (see Figure 4, Level II), as students give sometimes FIN and sometimes INF answers; finally it regroups into a more coherent, but still fragmented view at Level III (Figure 5). Indeed, Level III students were rather consistently on the "infinite side," but only 79 out of these 204 students (see Table 8) provided an Inf answer across all types of numbers.

Third, the results suggest that, despite the inconsistency revealed in students' responses, still certain patterns can be traced that suggest a possible developmental progression. It appears that awareness of the infinity of intermediate numbers comes first for intervals defined by natural numbers. Students gave more sophisticated answers in the case of natural as compared to non-natural numbers, a fact that was reflected in the significant performance differences in the respective items. In addition, more sophisticated answers for natural numbers appeared earlier in terms of age. In general, INF answers for natural numbers were given even by students who took a "finiteness" position in the remaining questions (i.e., the students at Level I). Our data also suggest that awareness of the infinity of numbers in an interval comes next for decimals: Students performed better in the decimals than in the fractions items, a difference that was significant in the case of the younger students (seventh graders). In addition, consistent INF* answers were more frequent in the case of decimals, than fractions. Finally, differences along the natural/non-natural distinction, as well as the decimal/fraction distinction are even more clear if we take into consideration students' profiles: About 16% of the students were consistent in answering differently for decimals, compared to fractions and about 9% answered consistently differently for natural, compared to non-natural numbers.

The differences among natural numbers, decimals, and fractions in the context of ordering (discrete/dense) support our hypothesis that they might be treated as different, unrelated sets of numbers. Students' difficulty to conceptualize the rational numbers as a unified, densely ordered, system of numbers, manifested itself in various forms in our findings, pointing to a class of synthetic conceptions of the rational numbers. Indeed, had a student understood that decimals and fractions are interchangeable representations of the same numbers and that natural numbers are a special case of decimals (or fractions), then awareness of the infinity of the intermediate numbers in one case should be transferred to the others. Furthermore, this understanding would allow for students to accept that the numbers belonging to an interval could take different forms.

Our findings indicate that all students have expanded their use of the term number to include non-natural numbers, and they have acquired relevant information. At the most naïve level of understanding (i.e., for students in the *All FIN** category), decimals and fractions inherit the discrete ordering of the natural numbers. This is true, despite the fact that students have acquired new knowledge about non-natural numbers, which they do employ in dealing with the "betweenness" tasks. One could ask, for example, how does a student reach the conclusion that there are *some* decimals between .01 and .02? Our experience from previous studies (Vamvakoussi & Vosniadou, 2004, 2007) is that students typically refer to .011, .012, . . . , .019, explaining that .01 and .02 can be expressed as .010 and .020. However, they do not repeat this process, presumably because of the constraint imposed by the idea of discreteness. But consider also that these students answer that there is a finite number of *decimals* between decimals, and a finite number of *fractions* between fractions. Thus, they may assign to both decimals and fractions the property of discreteness but they nevertheless keep them separated.

Then consider the students who believe that natural numbers behave differently with respect to ordering as compared to non-natural numbers. These students' initial theory of number is modified, allowing them to accept that between two numbers that were successive before, now there can be inserted new, infinitely many, non-natural numbers. However, at least in the context of order, natural and non-natural numbers are not deemed instances of rational numbers, expected to behave similarly. If we consider students' experiences both in and out of school settings (e.g., with the use of money, measurement of length or weight, the ruler, the number line), a possible interpretation of this finding could be that students view non-natural numbers as parts of whole numbers, or whole numbers plus/minus some part. This could allow for accommodating new information about non-natural numbers, without deposing natural numbers from their privileged position as the primary numbers, thus without the need to construct a new category of number, to which both natural and non-natural numbers are subsumed.

Then there were also the students who were found to be affected by the decimal/fraction distinction with respect to order (dense/discrete). It is possible that explicit information provided in the classroom, or more apt employment of procedural knowledge learned at school, may lead students to judge that there are infinitely many numbers between decimals, or even fractions. It appears, however, that this inference does not necessarily transfer from decimals to fractions, or vice versa.

Finally, even students who refer to the infinity of intermediate numbers across all types of numbers were reluctant to accept that these can be of different symbolic representations. Here we refer to the students placed in the *All INF** category, but not in the *Sophisticated* subcategory. It appears that even these students, despite their quite sophisticated answers with respect to the number of intermediate numbers, may not necessarily use the term "number" to refer to a unified system of numbers, invariant across representations.

Overall, the tendency to place only decimals between decimals and only fractions between fractions was particularly strong, across all grades (see Figure 2). Even in the case of natural numbers, where students were more apt to accept the infinity of numbers in the interval, there was still a considerable number of answers excluding fractions as possible intermediate numbers.

On the basis of the aforementioned considerations, we argue that students enter instruction with an initial theory of number as natural number, which gets modified, as they are exposed to instruction about decimals and fractions. The rational number perspective, from which decimals and fractions have the same referent and natural and non-natural numbers are members of the same family, is not yet (and should not be expected to be) available to students. Although new knowledge is acquired, the necessary interconnections are not made overnight. The view of rational numbers as consisting of different, unrelated sets of numbers can provide an intermediate state of illusionary coherence, which may allow for some students to account for the salient differences in form, behavior, and contexts of use, and move on in the knowledge acquisition process. In the context of ordering, this particular view of rational numbers is exhibited in a class of synthetic conceptions that share the family resemblance of assigning different behavior to natural numbers, decimals or fractions, in terms of the number or the type of intermediate numbers.

To sum up, the results of this study were in line with the predictions of our theoretical framework and provided information about the development of students' ideas regarding the density of rational numbers. Of course, these results are framed in the context of ordering, which is only one context in which numbers may be seen as different or the same. It can also be argued that the same students, or at least some of them, would be able to make reasonable assertions about the equivalence between fractions and decimals, depending on the affordances of the particular context (e.g., familiarity with the particular numbers involved), or a salient possibility to apply procedures they have been trained to use in school settings. Nevertheless, we believe that pointing out to a context in which students actually do not employ what they have extensively practiced at school, highlights conceptual difficulties with the rational concept that are persistent and not addressed by current instruction.

Indeed, age and level of instruction were found to have a positive impact on students' responses about the infinity of numbers in an interval: Ninth and eleventh graders performed better than seventh graders and they also gave less frequently the most naïve answer, namely that there is no other number between two rational numbers. However, no significant performance differences were found between ninth and eleventh graders, which suggests that further instruction about rational numbers does not address certain aspects of students' rational numbers thinking (e.g., the idea of discreteness of numbers). Indeed, this idea was found to be robust for ninth, but also for eleventh, graders. In addition, more ninth than eleventh graders were placed in the *Sophisticated* category. It is possible that this finding could be explained by the fact that ninth graders had been taught about basic properties of rational numbers more recently than eleventh graders, which in turn suggests that these gains may be transient. Moreover, the effect of the symbolic representation of the numbers involved was present in all age groups, which suggests that instruction does not support students to build the interconnections between them, despite the fact that starting at primary school they practice extensively in converting numbers from one form to the other. These considerations seem more serious in view of the fact that our younger participants, namely the seventh graders, had already been exposed to almost four years of instruction about rational numbers.

Limitations of the Present Study and Suggestions for Further Research

The present study attempted to account for certain aspects of the development of the rational number concept, focusing on students' understanding of the dense ordering of rational numbers. This was a cross-sectional study and, as such, was subject to inherent limitations relative to understanding learning at the individual level. Thus, although it provided information about the intermediate levels of understanding of the notion of density, there are important questions that cannot be immediately answered by this study. More specifically, this study does not shed direct light on whether individual students necessarily pass through all intermediate levels; whether they might stop progressing at one of those, or even regress to prior positions; how difficult and time consuming it is to move from a less to a more sophisticated position; and whether all shifts are equivalent in this respect. In addition, this study did not provide information regarding students' own explanations of their judgments, and the degree of their commitment to them in view of a different explanation or a different opinion. The aforementioned suggest that a different method is required in order to gain better insight into the individual's trajectory of conceptual change, possibly within the *microgenetic analyses of learning* paradigm (Siegler, 2006).

Also, as already pointed out, this study examined the issue of order only. A study with a wider range of tasks addressing more aspects of students' numbers (e.g., how numbers behave with respect to operations) could provide a more integrated picture of the process of development of the number concept. An interesting perspective would be to analyze, as Smith (1995) has done in a similar context, the reasoning and the strategies adopted by the students who are successful in this respect.

Further research might also be oriented toward instruction. A line of research worth following would be to investigate the contexts and conditions that are supportive in terms of students' conceptualization of rational numbers as a unified, densely ordered set. Yet another direction would be to investigate whether explicit teaching about density would be profitable in terms of rational and real number understanding.

Some Further Comments Regarding Instruction

Mathematics education researchers have acknowledged that students' difficulties with rational numbers can be attributed, to a large extent, to inadequate instruction. For example, Moss and Case (1999), summarizing the relevant considerations, have pointed out that instruction often emphasizes procedural versus conceptual knowledge of number; does not make the differences between natural and rational numbers explicit to students; and does not pay enough attention to students' own attempts to make sense of rational numbers and their notation, which is considered "transparent" (i.e., accessible to students with not much further explanation). Practically all the aforementioned considerations are relevant to our findings.

To be effective, our results suggest that rational number instruction needs to consider seriously that students' attempts to make sense of rational numbers, their properties, notation, and operations are heavily influenced by their prior knowledge of number. Thus, the assumptions we make about the organization of such prior knowledge makes a difference as to the planning and implementation of instruction. Thinking of students' prior knowledge of number in terms of a relatively coherent network of interrelated beliefs about the ontology and behavior of number, which needs to be restructured, could account to a large extent for the fact that mastering the rational number concept is gradual, time consuming, and difficult to achieve.

Dealing with this issue requires curricula designers, but also teachers, to take a long-term perspective on the development of mathematical knowledge, which includes the anticipation of expansions of meaning, and purposeful bridging between students initial ideas and the intended mathematical meanings (Greer, 2006). We believe that the framework theory approach to conceptual change might contribute in providing mathematics teachers, as well as curricula designers, with information necessary to anticipate students' conceptual difficulties and gain some insight into the intermediate states of understanding. In this direction, it is also necessary to acknowledge that building on students' prior knowledge does not always facilitate learning; on the contrary, in some cases it may in fact have an adverse effect (see also Resnick, 2006). For example, it is widely documented that the part-whole metaphor is popular as a construct appropriate to introduce fractions, because it builds on students' natural number knowledge and is thus considered more accessible to them. This is because the parts of a whole can be treated like discrete objects and can be subject to natural number manipulations, like counting. However, over-emphasis on the part–whole construct hinders students' rational number reasoning in the long run (Mamede et al., 2005; Moss, 2005).

There have been a variety of proposals about when rational number reasoning should be introduced in the curriculum, as well as the basis upon which it should be built. These include introducing instruction of non-natural numbers earlier in the curriculum, or even simultaneously with natural numbers, and build upon the notion of equal sharing or measurement (see Ni & Zhou, 2005 for an extensive discussion), and also splitting and perceiving proportionality (e.g. Moss & Case, 1999; Smith et al., 2005). In light of our results and our interpretation of them, we would like to note that the measurement aspect of rational number provides the opportunity to build the number concept on the same basis for natural and non-natural numbers. Moreover, measurement can be gradually associated with the (continuous) number line, a representational tool that can support students to conceptualize natural and non-natural numbers as members of the same family, and to understand that the different symbolic representations of a number actually refer to the same mathematical object (see also Kilpatrick et al., 2001). However, careful, long-term, planned investigation is needed for the viability and value of this suggestion to be tested.

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