# Chapter 16
# An Introduction to Design-Based Research with an Example From Statistics Education

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**Abstract** This chapter arose from the need to introduce researchers, including Master and PhD students, to design-based research (DBR). In Sect. 16.1 we address key features of DBR and differences from other research approaches. We also describe the meaning of validity and reliability in DBR and discuss how they can be improved. Section 16.2 illustrates DBR with an example from statistics education.

**Keywords** Design based research • Statistics education

## 16.1 Theory of Design-Based Research

### *16.1.1 Purpose of the Chapter*

The purpose of this chapter is to introduce researchers, including Master and PhD students, to design-based research. In our research methods courses for this audience and in our supervision of PhD students, we noticed that students considered key publications in this field unsuitable as introductions. These publications have mostly been written to inform or convince established researchers who already have considerable experience with educational research. We therefore see the need to write for an audience that does not have that level of experience, but may want to know about design-based research. We do assume a basic knowledge of the main research approaches (e.g., survey, experiment, case study) and methods (e.g., interview, questionnaire, observation).

Compared to other research approaches, educational design-based research (DBR) is relatively new (Anderson and Shattuck 2012). This is probably the reason that it is not discussed in most books on qualitative research approaches. For example, Creswell (2007) distinguishes five qualitative approaches, but these do not include DBR (see also Denscombe 2007). Yet DBR is worth knowing about, espe-

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A. Bikner-Ahsbahs et al. (eds.), *Approaches to Qualitative Research in Mathematics Education*, Advances in Mathematics Education,
DOI 10.1007/978-94-017-9181-6_16

cially for students who will become teachers or researchers in education: Design-based research is claimed to have the potential to bridge the gap between educational practice and theory, because it aims both at developing theories about domain-specific learning and the means that are designed to support that learning. DBR thus produces both useful products (e.g., educational materials) and accompanying scientific insights into how these products can be used in education (McKenney and Reeves 2012; Van den Akker et al. 2006). It is also said to be suitable for addressing complex educational problems that should be dealt with in a holistic way (Plomp and Nieveen 2007).

In line with the other chapters in this book, Sect. 16.1 provides a general theory of the research approach under discussion and Sect. 16.2 gives an example from statistics education on how the approach can be used.

## *16.1.2 Characterizing Design-Based Research*

In this section we outline some characteristics of DBR, compare it with other research approaches, go over terminology and history, and finally summarize DBR's key characteristics.

### 16.1.2.1 Integration of Design and Research

Educational design-based research (DBR) can be characterized as research in which the design of educational materials (e.g., computer tools, learning activities, or a professional development program) is a crucial part of the research. That is, the design of learning environments is interwoven with the testing or developing of theory. The theoretical yield distinguishes DBR from studies that aim solely at designing educational materials through iterative cycles of testing and improving prototypes.

A key characteristic of DBR is that educational ideas for student or teacher learning are formulated in the design, but can be adjusted during the empirical testing of these ideas, for example if a design idea does not quite work as anticipated. In most other interventionist research approaches design and testing are cleanly separated. See further the comparison with a randomized controlled trial in Sect. 16.1.2.5.

### 16.1.2.2 Predictive and Advisory Nature of DBR

To further characterize DBR it is helpful to classify research aims in general (cf. Plomp and Nieveen 2007):

- To describe (e.g., What conceptions of sampling do seventh-grade students have?)
- To compare (e.g., Does instructional strategy A lead to better test scores than instructional strategy B?)

- To evaluate (e.g., How well do students develop an understanding of distribution in an instructional sequence?)
- To explain or to predict (e.g., Why do so few students choose a bachelor in mathematics or science? What will students do when using a particular software package?)
- To advise (e.g., How can secondary school students be supported to learn about correlation and regression?)

Many research approaches such as surveys, correlational studies, and case studies, typically have descriptive aims. Experiments often have a comparative aim, even though they should in Cook's (2002) view "be designed to *explain* the consequences of interventions and not just to describe them" (p. 181, emphasis original). DBR typically has an explanatory and advisory aim, namely to give theoretical insights into how particular ways of teaching and learning can be promoted. The type of theory developed can also be of a predictive nature: Under conditions X using educational approach Y, students are likely to learn Z (Van den Akker et al. 2006).

Research projects usually have one overall aim, but several stages of the project can have other aims. For example, if the main aim of a research project is to advise how a particular topic (e.g., sampling) should be taught, the project most likely has parts in which phenomena are described or evaluated (e.g., students' prior knowledge, current teaching practices). It will also have a part in which an innovative learning environment has to be designed and evaluated before empirically grounded advice can be given. This implies that research projects are layered. Design-based research (DBR) has an overall predictive or advisory aim but often includes research stages with a descriptive, comparative, or evaluative aim.

#### 16.1.2.3 The Role of Hypotheses and the Engineering Nature of DBR

In characterizing DBR as different from other research approaches, we also need to address the role of hypotheses in theory development. Put simply, a scientific theory can explain particular phenomena and predict what will happen under particular conditions. When developing or testing a theory, scientists typically use hypotheses—conjectures that follow from some emergent theory that still needs to be tested empirically. This means that hypotheses should be formulated in a form in which they can be verified or falsified. The testing of hypotheses is typically done in an experiment: Reality is manipulated according to a theory-driven plan. If hypotheses are confirmed, this is support for the theory under construction.

Just as in the natural sciences, it is not always possible to test hypotheses empirically within a short period of time. As a starting point design researchers, just like many scientists in other disciplines, use thought experiments—thinking through the consequences of particular ideas. When preparing an empirical teaching experiment, design researchers typically do a thought experiment on how teachers or students will respond to particular tools or tasks based on their practical and theoretical knowledge of the domain (Freudenthal 1991).

In empirical experiments, a hypothesis is formulated beforehand. A theoretical idea is operationalized by designing a particular setting in which only this particular feature is isolated and manipulated. To stay objective experimental researchers are often not present during the interventions. In typical cases, they collect only pre- and posttest scores. In design-based research, however, researchers continuously take their best bets (Lehrer and Schauble 2001), even if this means that some aspect of the learning environment during or after a lesson has to be changed. In many examples, researchers are involved in the teaching or work closely with teachers or trainers to optimize the learning environment (McClain and Cobb 2001; Smit and Van Eerde 2011; Hoyles et al. 2010). In the process of designing and improving educational materials (which we take as a prototypical case in this chapter), it does not make sense to wait until the end of the teaching experiment before changes can be made. This would be inefficient.

DBR is therefore sometimes characterized as a form of didactical engineering (Artigue, 1988): didactical engineering: Something has to be made with whatever theories and resources are available. The products of DBR are judged on innovativeness and usefulness, not just on the rigor of the research process that is more prominent in evaluating true experiments (Plomp 2007).

In many research approaches, changing and understanding a situation are separated. However, in design-based research these are intertwined in line with the following adage that is also common in sociocultural traditions: If you want to understand something you have to change it, and if you want to change something you have to understand it (Bakker 2004a, p. 37).

#### 16.1.2.4 Open and Interventionist Nature of DBR

Another way to characterize DBR is to contrast it with other approaches on the following two dimensions: naturalistic vs. interventionist and open vs. closed. Naturalistic studies analyze how learning takes place without interference by a researcher. Examples of naturalistic research approaches are ethnography and surveys. As the term suggests, interventionist studies intervene in what naturally happens: Researchers deliberately manipulate a condition or teach according to particular theoretical ideas (e.g., inquiry-based or problem-based learning). Such studies are necessary if the type of learning that researchers want to investigate is not present in naturalistic settings. Examples of interventionist approaches are experimental research, action research, and design-based research.

Research approaches can also be more open or closed. The term *open* here refers to little control of the situation or data whereas *closed* refers to a high degree of control or a limited number of options (e.g., multiple choice questions). For example, surveys by means of questionnaires with closed questions or responses on a Likert scale are more closed than surveys by means of semi-structured interviews. Likewise, an experiment comparing two conditions is more closed than a DBR project in which the educational materials or ways of teaching are emergent and adjustable. Different research approaches can thus be positioned in a two-by-two table as in Table 16.1. DBR thus shares an interventionist nature with experiments and action research. We therefore continue by comparing DBR with experiments (16.1.2.5) and with action research (16.1.2.6).

**Table 16.1** Naturalistic vs. interventionist and open vs. closed research approaches

| | Naturalistic | Interventionist |
|---|---|---|
| Closed | Survey: questionnaires with closed questions | Experiment (randomized controlled trial) |
| Open | Survey: interviews with open questions | Action research |
| | Ethnography | Design-based research |

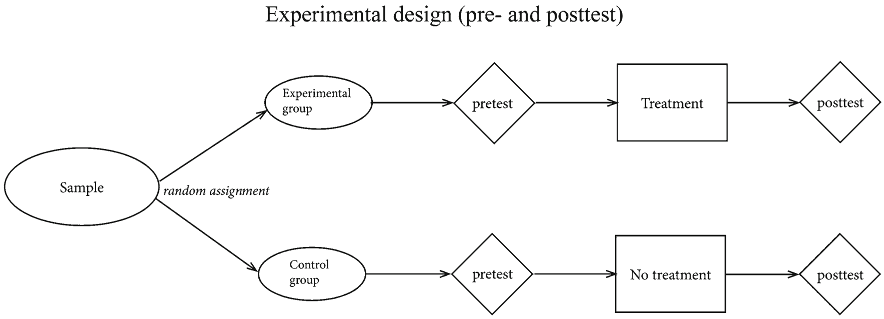



**Fig. 16.1** A pre-posttest experimental design (randomized controlled trial)

#### 16.1.2.5 Comparison of DBR with Randomized Controlled Trials (RCT)

A randomized controlled trial (RCT) is sometimes referred to as "true" experiment. Assume we want to know whether a new teaching strategy for a particular topic in a particular grade is better than the traditionally used one. To investigate this question one could randomly assign students to the experimental (new teaching strategy) or control condition (traditional strategy), measure performances on pre- and posttests, and use statistical methods to test the null hypothesis that there is no significant difference between the two conditions. The researchers' hope is that this hypothesis can be rejected so that the new type of intervention (informed by a particular theory) proves to be better. The underlying rationale is: If we know "what works" we can implement this method and have better learning results (see Fig. 16.1).

This so-called experimental approach of randomized controlled trials (Creswell 2005) is sometimes considered the highest standard of research (Slavin 2002). It has a clear logic and is a convincing way to make causal and general claims about what works. It is based on a research approach that has proven extremely helpful in the natural sciences.

However, its limitations for education are discussed extensively in the literature (Engeström 2011; Olsen 2004). Here we mention two related arguments. First, if we know what works, we still do not know why and when it works. Even if the new strategy is implemented, it might not work as expected because teachers use it in less than optimal ways.

An example can clarify this. When doing research in an American school, we heard teachers complain about their managers' decision that every teacher had to start every lesson with a warm-up activity (e.g., a puzzle). Apparently it had been proven by means of an RCT that student scores were significantly higher in the experimental condition in which lessons started with a warm-up activity. The negative effect in teaching practice, however, was that teachers ran out of good ideas for warm-up activities, and that these often had nothing to do with the topic of the lesson. Effectively, teachers therefore lost five minutes of every lesson. Better insight into how and why warm-up activities work under particular conditions could have improved the situation, but the comparative nature of RCT had not provided this information because only the variable of starting the lesson with or without warm-up activity had been manipulated.

A second argument why RCT has its limitations is that a new strategy has to be designed before it can be tested, just like a Boeing airplane cannot be compared with an Airbus without a long tradition of engineering and producing such airplanes. In many cases, considerable research is needed to design innovative approaches. Design-based research emerged as a way to address this need of developing new strategies that could solve long-standing or complex problems in education.

Two discussion points in the comparison of DBR and RCT are the issues of generalization and causality. The use of random samples in RCT allows generalization to populations, but in most educational research random samples cannot be used. In response to this point, researchers have argued that theory development is not just about populations, but rather about propensities and processes (Frick 1998). Hence rather than generalizing from a random sample to a population (statistical generalization), many (mainly qualitative) research approaches aim for generalization to a theory, model or concept (theoretical or analytic generalization) by presenting findings as particular cases of a more general model or concept (Yin 2009).

Where the use of RCTs can indicate the intervention or treatment being the cause of better learning, DBR cannot claim causality with the same convincing rigor. This is not unique to DBR: All qualitative research approaches face this challenge of drawing causal claims. In this regard it is helpful to distinguish two views on causality: a regularity, variance-oriented understanding of causality versus a realist, process-oriented understanding of causality (Maxwell 2004). People adopting the first view think that causality can only be proven on the basis of regularities in larger data sets. People adopting the second view make it plausible on the basis of circumstantial evidence of observed processes that what happened is most likely caused by the intervention (e.g., Nathan and Kim 2009). The first view is underlying the logic of RCT: If we randomly assign subjects to an experimental and control condition, treat only the experimental group and find a significant difference between the two groups, then it can only be attributed to the difference in condition (the treatment). However, if we were to adopt the same regularity view on causality we would never be able to identify the cause of singular events, for example why a driver hit a tree. From the second, process-oriented view, if a drunk driver hits a tree we can judge the circumstances and judge it plausible that his drunkenness was an important explanation because we know that alcohol can cause less control, slower reaction time et cetera. Similarly, explanations for what happens in classrooms should be possible according to a process-oriented position based on what happens in response to particular interventions. For example, particular student utterances are very unlikely if not deliberately fostered by a teacher (Nathan and Kim 2009). Table 16.2 summarizes the main points of the comparison of RCT and DBR.

#### 16.1.2.6 Comparison of DBR with Action Research

Like action research, DBR typically is interventionist and open, involves a reflective and often cyclic process, and aims to bridge theory and practice (Opie 2004). In both approaches the teacher can be also researcher. In action research, the researcher is not an observer (Anderson and Shattuck 2012), whereas in DBR s/he can be observer. Furthermore, in DBR design is a crucial part of the research, whereas in action research the focus is on action and change, which can but need not involve the design of a new learning environment. DBR also more explicitly aims for instructional theories than does action research. These points are summarized in Table 16.3.

**Table 16.2** Comparison of experimental versus design-based research

| Experiment (RCT) | Design-based research (DBR) |
|---|---|
| Testing theory | Developing and testing theory simultaneously |
| Comparison of existing teaching methods by means of experimental and control groups | Design of an innovative learning environment long |
| Proof of what works | Insight into how and why something works |
| Research interest is isolated by manipulating variables separately | Holistic approach long white word |
| Statistical generalization | Analytic or theoretical generalization, transferability to other situations |
| Causal claims based on a regularity view on causality are possible | Causality should be handled with great care and be based on a realist, process-oriented view on causality |

**Table 16.3** Commonalities and differences between DBR and action research

| | DBR | Action research |
|---|---|---|
| Commonalities | Open, interventionist, researcher can be participant, reflective cyclic process | |
| Differences | Researcher can be observer | Researcher can only be participant |
| | Design is necessary | Design is possible |
| | Focus on instructional theory | Focus on action and improvement of a situation |

#### 16.1.2.7 Names and History of DBR

In its relatively brief history, DBR has been presented under different names. *Design-based research* is the name used by the Design-Based Research Collective (see special issues in Educational Researcher, 2003; Educational Psychologist 2004; Journal of the Learning Sciences 2004). Other terms for similar approaches are:

- Developmental or development research (Freudenthal 1988; Gravemeijer 1994; Lijnse 1995; Romberg 1973; Van den Akker 1999)
- Design experiments or design experimentation (Brown 1992; Cobb et al. 2003a; Collins 1992)
- Educational design research (Van den Akker et al. 2006)

The reasons for these different terms are mainly historical and rhetorical. In the 1970s Romberg (1973) used the term *development research* for research accompanying the development of curriculum. Discussions on the relation between research and design in mathematics education, especially on didactics, mainly took place in Western Europe in the 1980s and the 1990s, particularly in the Netherlands (e.g., Freudenthal 1988; Goffree 1979), France (e.g., Artigue 1988, cf. Artigue Chap. 17) and Germany (e.g., Wittmann 1992). The term *developmental research* is a translation of the Dutch *ontwikkelingsonderzoek*, which Freudenthal introduced in the 1970s to justify the development of curricular materials as belonging to a university institute (what is now called the Freudenthal Institute) because it was informed by and leading to research on students' learning processes (Freudenthal 1978; Gravemeijer and Koster 1988; De Jong and Wijers 1993). The core idea was that development of learning environments and the development of theory were intertwined. As Goffree (1979, p. 347) put it: "Developmental research in education as presented here, shows the characteristics of both developmental and fundamental research, which means aiming at new knowledge that can be put into service in continued development." At another Dutch university (Twente University), the term *ontwerpgericht* (design-oriented) research was more common, but there the focus was more on the curriculum than on theory development (Van den Akker 1999). One disadvantage of the terms 'development' and 'developmental' is their connotations to developmental psychology and research on children's development of concepts. This might be one reason that this term is hardly used anymore.

In the United States, the terms *design experiment* and *design research* were more common (Brown 1992; Cobb et al. 2003a; Collins 1992; Edelson 2002). One advantage of these terms is that design is more specific than development. One possible disadvantage of the term *design experiment* can be explained by reference to a critical paper by Paas (2005) titled *Design experiment: Neither a design nor an experiment*. The confusion that his pun refers to is two-fold. First, in many educational research communities the term *design* is reserved for research design (e.g., comparing an experimental with a control group), whereas the term in design research refers to the design of learning environments (Sandoval and Bell 2004). Second, for many researchers, also outside the learning sciences, the term *experiment* is reserved for "true" experiments or RCTs. In design experiments, hypotheses certainly play an important role, but they are not fixed and tested once. Instead they may be emergent, multiple, and temporary. In line with the Design-Based Research Collective, we use the term *design-based research* because this suggests that it is predominantly research (hence leading to a knowledge claim) that is based on a design process.

#### 16.1.2.8 Theory Development in Design-Based Research

We have already stated that theory typically has a more central role in DBR than in action research. To address the role of theory in DBR, it is helpful to summarize diSessa and Cobb's (2004) categorization of different types of theories involved in educational research. They distinguish:

- Grand theories (e.g., Piaget's phases of intellectual development; Skinner's behaviorism)
- Orienting frameworks (e.g., constructivism, semiotics, sociocultural theories)
- Frameworks for action (e.g., designing for learning, Realistic Mathematics Education)
- Domain-specific theories (e.g., how to teach density or sampling)
- Hypothetical Learning Trajectories (Simon 1995) or didactical scenarios (Lijnse 1995; Lijnse and Klaassen 2004) formulated for specific teaching experiments (explained in Sect. 16.1.3).

As can be seen from this categorization, there is a hierarchy in the generality of theories. Because theories developed in DBR are typically tied to specific learning environments and learning goals, they are humble and hard to generalize. Similarly, it is very rare that a theoretical contribution to aerodynamics will be made in the design of an airplane; yet innovations in airplane design occur regularly. The use of grand theoretical frameworks and frameworks for action is recommended, but researchers should be careful to manage the gap between the different types of theory on the one hand and design on the other (diSessa and Cobb 2004). If handled with care, DBR can then provide the basis for refining or developing theoretical concepts such as meta-representational competence, sociomathematical norms (diSessa and Cobb), or whole-class scaffolding (Smit et al. 2013).

#### 16.1.2.9 Summary of Key Characteristics of Design-Based Research

So far we have characterized DBR in terms of its predictive and advisory aim, particular way of handling hypotheses, its engineering nature and differences from other research methods. Here we summarize five key characteristics of DBR as identified by Cobb et al. (2003a):

1. The first characteristic is that its purpose is *to develop theories about learning and the means that are designed to support that learning*. In the example provided in Sect. 16.2 of in this chapter, Bakker (2004a) developed an instruction theory for early statistics education and instructional means (e.g. computer tools

and accompanying learning activities) that support the learning of a multifaceted notion of statistical distribution.

2. The second characteristic of DBR is its *interventionist* nature. One difference with RCTs is that interventions in the DBR tradition often have better ecological validity—meaning that learning already takes place in learning ecologies as they occur in schools and thus methods measure better what researchers want to measure, that is learning in natural situations. Findings from experiments do not have to be translated as much from controlled laboratory situations to the less controlled ecology of schools or courses. In technical terms, theoretical products of DBR "have the potential for rapid pay-off because they are filtered in advance for instrumental effect" (Cobb et al. 2003a, p. 11).
3. The third characteristic is that DBR has *prospective and reflective components* that need not be separated by a teaching experiment. In implementing hypothesized learning (the prospective part) the researchers confront conjectures with actual learning that they observe (reflective part). Reflection can be done after each lesson, even if the teaching experiment is longer than one lesson. Such reflective analysis can lead to changes to the original plan for the next lesson. Kanselaar (1993) argued that any good educational research has prospective and reflective components. As explained before, however, what distinguishes DBR from other experimental approaches is that in DBR these components are not separated into the formulation of hypotheses before and after a teaching experiment.
4. The fourth characteristic is the *cyclic* nature of DBR: Invention and revision form an iterative process. Multiple conjectures on learning are sometimes refuted and alternative conjectures can be generated and tested. The cycles typically consist of the following phases: preparation and design phase, teaching experiment, and retrospective analysis. These phases are worked out in more detail later in this chapter. The results of such a retrospective analysis mostly feed a new design phase. Other types of educational research ideally also build upon prior experiments and researchers iteratively improve materials and theoretical ideas in between experiments but in DBR changes can take place during a teaching experiment or series of teaching experiments.
5. The fifth characteristic of DBR is that the *theory* under development *has to do real work*. As Lewin (1951, p. 169) wrote: "There is nothing so practical as a good theory." Theory generated from DBR is typically humble in the sense that it is developed for a specific domain, for instance statistics education. Yet it must be general enough to be applicable in different contexts such as classrooms in other schools in other countries. In such cases we can speak of transferability.

### *16.1.3 Hypothetical Learning Trajectory (HLT)*

DBR typically consists of cycles of three phases each: preparation and design, teaching experiment, and retrospective analysis. One might argue that the term 'retrospective analysis' is pleonastic: All analysis is in retrospect, after a teaching experiment. However, we use it here to distinguish it from analysis on the fly, which takes place during a teaching experiment, often between lessons.

A design and research instrument that proves useful during all phases of DBR is the *hypothetical learning trajectory* (HLT), which we regard as an elaboration of Freudenthal's thought experiment. Simon (1995) defined the HLT as follows:

> The hypothetical learning trajectory is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. (p. 136)

Simon used the HLT for one or two lessons. Series of HLTs can be used for longer sequences of instruction (also see the literature on didactical scenarios in Lijnse 1995). The HLT is a useful research instrument to manage the gap between an instruction theory and a concrete teaching experiment. It is informed by general domain-specific and conjectured instruction theories (Gravemeijer 1994), and it informs researchers and teachers how to carry out a particular teaching experiment. After the teaching experiment, it guides the retrospective analysis, and the interplay between the HLT and empirical results forms the basis for theory development. This means that an HLT, after it has been mapped out, has different functions depending on the phase of the DBR and continually develops through the different phases. It can even change during a teaching experiment.

#### 16.1.3.1 HLT in the Design Phase

The development of an HLT starts with an analysis of how the mathematical topic of the design study is elaborated in the curriculum and the mathematical textbooks, an analysis of the difficulties students encounter with this topic, and a reflection on what they should learn about it. These analyses result in the formulation of provisional mathematical learning goals that form the orientation point for the design and redesign of activities in several rounds. While designing mathematical activities the learning goals may become better defined. During these design processes the researcher also starts formulating hypotheses about students' potential learning and about how the teacher would support students' learning processes. The confrontation of a general rationale with concrete tasks often leads to a more specific HLT, which means that the HLT gradually develops during the design phase (Drijvers 2003).

An elaborated HLT thus includes mathematical learning goals, students' starting points with information on relevant pre-knowledge, mathematical problems and assumptions about students' potential learning processes and about how the teacher could support these processes.

#### 16.1.3.2 HLT in Teaching Experiment

During the teaching experiment, the HLT functions as a guideline for the teacher and researcher for what to focus on in teaching, interviewing, and observing. It may happen that the teacher or researcher feels the need to adjust the HLT or instructional activity for the next lesson. As Freudenthal wrote (1991, p. 159), the cyclic alternation of research and development can be more efficient the shorter the cycle is. Minor changes in the HLT are usually made because of incidents in the classroom such as student strategies that were not foreseen, activities that were too difficult, and so on. Such adjustments are generally not accepted in comparative experimental research, but in DBR, changes in the HLT are made to create optimal conditions and are regarded as elements of the data corpus. This means that these changes have to be reported well and the information is stronger when changes are supported by theoretical considerations. The HLT can thus also change during the teaching experiment phase.

#### 16.1.3.3 HLT in the Retrospective Analysis

During the retrospective analysis, the HLT functions as a guideline determining what the researcher should focus on in the analysis. Because predictions are made about students' learning, the researcher can contrast those conjectures with the observations made during the teaching experiment. Such an analysis of the interplay between the evolving HLT and empirical observations forms the basis for developing an instruction theory. After the retrospective analysis, the HLT can be reformulated, often more drastically than during the teaching experiment, and the new HLT can guide a subsequent design phase.

An HLT can be seen as a concretization of an evolving domain-specific instruction theory. Conversely, the instruction theory is informed by evolving HLTs. For example, if patterns of an HLT stabilize after a few cycles, these generalized patterns in learning or instruction and the insights of how these patterns are supported by instructional means can become part of the emerging instruction theory.

Overall, the idea behind developing an HLT is not to design the perfect instructional sequence, which in our view does not exist, but to provide empirically grounded results that others can adjust to their local circumstances. The HLT remains hypothetical because each situation, each teacher, and each class is different. Yet patterns can be found in students' learning that are similar across different teaching experiments. Those patterns and the insights of how particular educational activities support students in particular kinds of reasoning can be the basis for a more general instructional theory of how a particular domain can be taught. Bakker (2004a), for example, noted that when estimating the number of elephants in a picture, students typically used one of four strategies, and these four strategies reoccurred in all of the five classrooms in which he used the same task. Having observed such a pattern in strategy use, the design researcher can assume the pattern to be an element of the instruction theory.

For some readers, the term 'trajectory' might have a linear connotation. Although we aim for a certain direction, like the course of a ship, Bakker's (2004a) HLTs were non-linear in the sense that he did not make a linear sequence of activities in advance that he strictly adhered to (cf. Fosnot and Dolk 2001). Moreover, two subtrajectories came together later on in the sequence. In the following sections we give a more detailed description of the three phases of a DBR cycle and discuss relevant methodological issues. Further details about hypothetical learning trajectories can be found in a special issue of *Mathematical Thinking and Learning* (Mathematical Thinking and Learning 2004, volume 6, issue 2) devoted to HLTs.

The term HLT stems from research in which the teacher was a researcher or a member of the research team (Simon 1995). However, if the teacher is not so familiar with the research team's intentions it may be necessary to pay extra attention to what the teacher can or should do to realize the potential of the learning activities. In such cases, the terms *hypothetical teaching and learning trajectory* (HTLT) or *teaching and learning strategy* (Dierdorp et al. 2011) may be more appropriate.

## *16.1.4 Phases in DBR*

### 16.1.4.1 Phase 1: Preparation and Design

It is evident that the relevant present knowledge about a topic should be studied first. Gravemeijer (1994) characterizes the design researcher as a tinkerer or, in French, a *bricoleur*, who uses all the material that is at hand, including theoretical insights and practical experience with teaching and designing.

In the first design phase, it is recommended to collect and invent a set of tasks that could be useful and discuss these with colleagues who are experienced in designing for mathematics education. An important criterion for selecting a task is its potential role in the HLT towards the mathematical end goal. Could it possibly lead to types of reasoning that students could build upon towards that end goal? Would it be challenging? Would it be a meaningful context for students?

There are several design heuristics, principles, and guidelines. In Sect. 16.2 we explain heuristics from the theory of Realistic Mathematics Education.

### 16.1.4.2 Phase 2: Teaching Experiment

The notion of a teaching experiment arose in the 1970s. Its primary purpose was to experience students' learning and reasoning first-hand, and it thus served the purpose of eliminating the separation between the practice of research and the practice of teaching (Steffe and Thompson 2000). Over time, teaching experiments proved useful for a broader purpose, namely as part of DBR. During a teaching experiment, researchers and teachers use activities and types of instruction that according to the HLT seem most appropriate at that moment. Observations in one lesson and theoretical arguments from multiple sources can influence what is done in the next lesson. Observations may include student or teacher deviations from the HLT.

Hence, this type of research is different from experimental research designs in which a limited number of variables are manipulated and effects on other variables are measured. The situation investigated here, the learning of students in a new context with new tools and new end goals, is too complicated for such a set-up.

Besides that, a different type of knowledge is looked for, as pointed out earlier in this chapter: We do not want to assess innovative material or a theory, but we need prototypical educational materials that could be tested and revised by teachers and researchers, and a domain-specific instruction theory that can be used by others to formulate their own HLTs suiting local contingencies.

During a teaching experiment, data collection typically includes student work, tests before and after instruction, field notes, audio recordings of whole-class discussions, and video recordings of every lesson and of the final interviews with students and teachers. We further find 'mini-interviews' with students, lasting from about twenty seconds to four minutes, very useful provided that they are carried out systematically (Bakker 2004a).

#### 16.1.4.3 Retrospective Analysis

We describe two types of analysis useful in DBR, a task oriented analysis and a more overall, longitudinal, cyclic approach. The first is to compare data on students' actual learning during the different tasks with the HLT. To this end we find the data analysis matrix (Table 16.4) described in Dierdorp et al. (2011) useful. The left part of the matrix summarizes the HLT and the right part is filled with excerpts from relevant transcripts, clarifying notes from the researcher as well as a quantitative impression of how well the match was between the assumed leaning as formulated in the HLT and the observed learning. With such analysis it is possible to give an overview, as in Table 16.5, which can help to identify problematic sections in the educational materials. Insights into why particular learning takes place or does not

**Table 16.4** Data analysis matrix for comparing HLT and actual learning trajectory (ALT)

| Hypothetical learning trajectory | | | Actual learning trajectory | | |
|---|---|---|---|---|---|
| Task number | Formulation of the task | Conjecture of how students would respond | Transcript excerpt | Clarification | Match between HLT and ALT: Quantitative impression of how well the conjecture and actual learning matched (e.g., –, 0, +) |

**Table 16.5** ALT result compared with HLT conjectures for the tasks involving a particular type of reasoning

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| + | | | | x | x | | | x | x | x | x | x | | x | x | x | x | x | x |
| ± | x | | x | | | | | | | | | | x | | | | | | |
| – | | x | | | | x | x | | | | | | | | | | | | |
| Task: | 5d | 5f | 6a | 6c | 7 | 8 | 9c | 9e | 10b | 11c | 15 | 17 | 23b | 23c | 24a | 24c | 25d | 34a | 42 |

*Note*: an x means how well the conjecture accompanying that task matched the observed learning (– refers to confirmation for up to 1/3 of the students, and + to at least 2/3 of the students) take place help to improve the HLTs in subsequent cycles of DBR. This iterative process allows the researcher to improve the predictive power of HLTs across subsequent teaching experiments.

An elaborated HLT would include assumptions about students' potential learning and about how the teacher would support students' learning processes. In this task-oriented analysis above no information is included about the role of the teacher. If there are crucial differences between students' assumed and observed learning processes or if the teaching has been observed to diverge radically from what the researcher had intended, the role of the teacher should be included into the analysis in search of explanations for these discrepancies.

A comparison of HLTs and observed learning is very useful in the redesign process, and allows answers to research questions that ask how particular learning goals could be reached. However, in our experience additional analyses are often needed to gain more theoretical insights into the learning process. An example of such additional analysis is a method inspired by the *constant comparative method* (Glaser and Strauss 1967; Strauss and Corbin 1998) and Cobb and Whitenack's (1996) method of longitudinal analyses. Bakker (2004a) used this type of analysis in his study in the following way. First, all transcripts were read and the videotapes were watched chronologically episode-by-episode. With the HLT and research questions as guidelines, conjectures about students' learning and views were generated and documented, and then tested against the other episodes and other data material (student work, field notes, tests). This testing meant looking for confirmation and counter-examples. The process of conjecture generating and testing was repeated. Seemingly crucial episodes were discussed with colleagues to test whether they agreed with our interpretation or perhaps could think of alternative interpretations. This process is called *peer examination*.

For the analysis of transcripts or videos it is worth considering computer software such as Atlas.ti (Van Nes and Doorman 2010) for coding the transcripts and other data sources. As in all qualitative research, data triangulation (Denscombe 2007) is commonly used in design-based research.

### *16.1.5 Validity and Reliability*

Researchers want to analyze data in a reliable way and draw conclusions that are valid. Therefore, validity and reliability are important concerns. In brief, validity concerns whether we really measure what we intend to measure. Reliability is about independence of the researcher. A brief example may clarify the distinction. Assume a researcher wants to measure students' mathematical ability. He gives everyone 7 out of 10. Is this a valid way of measuring? Is this a reliable way?

It is a very reliable way because the instruction "give all students a 7" can be reliably carried out, independently of the researcher. However, it is not valid, because there is most likely variation between students' mathematical ability, which is not taken into account with this way of measuring.

We should emphasize that validity and reliability are complex concepts with multiple meanings in different types of research. In qualitative research the meanings of validity and reliability are slightly different than in quantitative research. Moreover, there are so many types of validity and reliability that we cannot address them all. In this chapter we have focused on those types that seemed most relevant to us in the context of DBR. The issues discussed in this section are inspired by guidelines of Maso and Smaling (1998) and Miles and Huberman (1994), who distinguish between internal and external validity and reliability.

#### 16.1.5.1 Internal Validity

Internal validity refers to the quality of the data and the soundness of the reasoning that has led to the conclusions. In qualitative research, this soundness is also labeled as *credibility* (Guba 1981). In DBR, several techniques can be used to improve the internal validity of a study.

- During the retrospective analysis conjectures generated and tested for specific episodes are tested for other episodes or by data triangulation with other data material, such as field notes, tests, and other student work. During this testing stage there is a search for counterexamples to the conjectures.
- The succession of different teaching experiments makes it possible to test the conjectures developed in earlier experiments in later experiments.

Theoretical claims are substantiated where possible with transcripts to provide a rich and meaningful context. Reports about DBR tend to be long due to the *thick descriptions* (Geertz 1973) required. For example, the paper by Cobb et al. (2003b) is 78 pages long!

#### 16.1.5.2 External Validity

External validity is mostly interpreted as the generalizability of the results. The question is how we can generalize the results from these specific contexts to be useful for other contexts. An important way to do so is by framing issues as instances of something more general (Cobb et al. 2003a; Gravemeijer and Cobb 2006). The challenge is to present the results (instruction theory, HLT, educational activities) in such a way that others can adjust them to their local contingencies.

In addition to generalizability as a criterion for external validity we mention *transferability* (Maso and Smaling 1998). If lessons learned in one experiment are successfully applied in other experiments, this is a sign of successful generalization. At the end of Sect. 16.2 we give an example of how a new type of learning activity was successfully enacted in a new research project in another country.

#### 16.1.5.3 Internal Reliability

Internal reliability refers to the degree of how independently of the researcher the data are collected and analyzed. It can be improved with several methods. Data collection by objective devices such as audio- and video registrations contribute to the internal reliability. During his retrospective analysis Bakker (2004a) ensured reliability by discussing the critical episodes, including those discussed in Sect. 16.2, with colleagues for peer examination. For measuring interrater reliability, the agreement among independent researchers, it is advised to calculate not only the percentage of agreement but also use Cohen's kappa or another measure that takes into account the probability of agreement by chance (e.g., Krippendorff's alpha). It is not necessary for a second coder to code all episodes, but ensure that a random sample should be of sufficient size: The larger the number of possible codes, the larger the sample required (Bakkenes et al. 2010; Cicchetti 1976). Note that the term internal reliability can also refer to the consistency of responses on a questionnaire or test, often measured with help of Cronbach's alpha.

#### 16.1.5.4 External Reliability

External reliability usually denotes replicability, meaning that the conclusions of the study should depend on the subjects and conditions, and not on the researcher. In qualitative research, replicability is mostly interpreted as virtual replicability. The research must be documented in such a way that it is clear how the research has been carried out and how conclusions have been drawn from the data. A criterion for virtual replicability is 'trackability' (Gravemeijer and Cobb 2006), 'traceability' (Maso and Smaling 1998), or transparency (Akkerman et al. 2008). This means that the reader must be able to track or trace the learning process of the researchers and to reconstruct their study: failures and successes, procedures followed, the conceptual framework used, and the reasons for certain choices must all be reported. In Freudenthal's words:

> Developmental research means: experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience. (1991, p. 161)

We illustrate the general characterization and description of DBR of Sect. 16.1 by an example of a design study on statistics education in Sect. 16.2.

## 16.2 Example of Design-Based Research

In this second section we illustrate the theory of design-based research (DBR) as outlined in Sect. 16.1 with an example from Bakker's (2004a, b) PhD thesis on DBR in statistics education. We briefly describe the aim and theoretical background of this DBR project and then focus on one design idea, that of growing samples, to illustrate how it is related to different layers of theory and how it was analyzed. Finally we discuss the issue of generalizability. In the appendix we provide a structure of a DBR project with examples from this Sect. 16.2.

### 16.2.1 Relevance and Aim

The background problem addressed in Bakker's (2004a) research on statistics education was that many stakeholders were dissatisfied with what and how students learned about statistics. For example, in many curricula there was a focus on computing arithmetic means and making bar charts (Friel et al. 2001). Moreover, there was very little knowledge about how to use innovative educational statistics software (cf. Biehler et al. 2013, for an historical overview).

To solve these practical problems, Bakker's (2004a) aim was to contribute to an empirically grounded instruction theory for early statistics education with new computer tools for the age group from 11 to 14. Such a theory should specify patterns in students' learning as well as the means supporting that learning in the domain of statistics education. Like Cobb et al. (2003b), Bakker (2004a) focused his research on the concept of distribution as a key concept in statistics. One problem is that students tend to see isolated data points instead of a data set as a whole (Bakker and Gravemeijer 2004; Konold and Higgins 2003). Yet statistics is about features of data sets, in particular distributions of samples. The selected learning goal was therefore that distribution had to become an object-like entity with which students could see data sets as an entity with characteristics.

### 16.2.2 Research Question

Bakker's initial research question was: How can students with little statistical background develop a notion of distribution? In trying to answer this question in grade 7, however, Bakker came to include a focus on other statistical key concepts such as data, center, and sampling because these are so intricately connected to that of distribution (Bakker and Derry 2011). The concept of distribution also proved hard for seventh-grade students. The initial research question was therefore reformulated for grade 8 as follows: How can coherent reasoning about distribution be promoted in relation to data, variability, and sampling in a way that is meaningful for students with little statistical background?

Our point here is that research questions can change during a research project. Indeed, the better and sharper your research question is in the beginning of the project, the better and more focused your data collection will be. However, our experience is that most DBR researchers, due to progressive insight, end up with slightly different research questions than they started with.

As pointed out in Sect. 16.1, DBR typically draws on several types of theories. Given the importance of graphical representations in statistics education, it made sense for Bakker to draw on semiotics as an orienting framework. He came to focus on semiotics, in particular Peirce's ideas on diagrammatic reasoning. The domain-specific theory of Realistic Mathematics Education proved a useful framework for action in the design process even though it had hardly been applied in statistics education.

### *16.2.3 Orienting Framework: Diagrammatic Reasoning*

The learning goal was that distribution would become an object-like entity. Theories on reification of concepts (Sfard and Linchevski 1992) and the relation between process and concept (cf. Tall et al. 2000, on *procept*) were drawn upon. One theoretical question unanswered in the literature was what the process nature of a distribution could be. It is impossible to make sense of graphs without having appropriate conceptual structures, and it is impossible to communicate about concepts without any representations. Thus, to develop an instruction theory it is necessary to investigate the relation between the development of the meaning of graphs and concepts. After studying several theories in this area, Bakker deployed Peirce's semiotic theory on diagrammatic reasoning (Bakker 2007; Bakker and Hoffmann 2005). For Peirce, a diagram is a sign that is meant to represent relations. Diagrammatic reasoning involves three steps:

1. The first step is to *construct* a diagram (or diagrams) by means of a representational system such as Euclidean geometry, but we can also think of diagrams in computer software or of an informal student sketch of statistical distribution. Such a construction of diagrams is supported by the need to represent the relations that students consider significant in a problem. This first step may be called *diagrammatization*.
2. The second step of diagrammatic reasoning is to *experiment* with the diagram (or diagrams). Any experimenting with a diagram is executed within a not necessarily perfect representational system and is a rule or habit-driven activity. Contemporary researchers would stress that this activity is situated within a practice. What makes experimenting with diagrams important is the rationality immanent in them (Hoffmann 2002). The rules define the possible transformations and actions, but also the constraints of operations on diagrams. Statistical diagrams such as dot plots are also bound by certain rules: a dot has to be put above its value on the $x$ axis and this remains true even if for instance the scale is changed. Peirce stresses the importance of doing something when thinking or reasoning with diagrams:

> Thinking in general terms is not enough. It is necessary that something should be DONE. In geometry, subsidiary lines are drawn. In algebra, permissible transformations are made. Thereupon the faculty of observation is called into play. (CP 4.233—CP refers to Peirce's collected papers, volume 4, section 233)

In the software used in this research, students can do something with the data points such as organizing them into equal intervals or four equal groups.

3. The third step is to observe the results of experimenting. We refer to this as the *reflection* step. As Peirce wrote, the mathematician observing a diagram "puts before him an icon by the observation of which he detects relations between the parts of the diagram other than those which were used in its construction" (Peirce 1976 III, p. 749). In this way he can "discover unnoticed and hidden relations among the parts" (Peirce CP 3.363; see also CP 1.383). The power of diagrammatic reasoning is that "we are continually bumping up against hard fact. We expected one thing, or passively took it for granted, and had the image of it in our minds, but experience forces that idea into the background, and compels us to think quite differently" (Peirce CP 1.324).

Diagrammatic reasoning, in particular the reflection step, is what can introduce the 'new'. New implications within a given representational system can be found, but possibly the need is felt to construct a new diagram that better serves its purpose.

### *16.2.4 Domain-Specific Framework for Action: Realistic Mathematics Education (RME)*

As pointed out by diSessa and Cobb (2004), grand theories and orienting frameworks do not tell the design researcher how to design learning environments. For this purpose, frameworks for action can be useful. Here we discuss Realistic Mathematics Education (RME).

Our research took place in the tradition of RME as developed over the last 40 years at the Freudenthal Institute (Freudenthal 1991; Gravemeijer 1994; Treffers 1987; van den Heuvel-Panhuizen 1996). RME is a theory of mathematics education that offers a pedagogical and didactical philosophy on mathematical learning and teaching as well as on designing educational materials for mathematics education. RME emerged from research and development in mathematics education in the Netherlands in the 1970s and it has since been used and extended, also in other countries.

The central principle of RME is that mathematics should always be meaningful to students. For Freudenthal, mathematics was an extension of common sense, a system of concepts and techniques that human beings had developed in response to phenomena they encountered. For this reason, he advised a so-called *historical phenomenology* of concepts to be taught, a study of how concepts had been developed in relation to particular phenomena. The insights from such a study can be input for the design process (Bakker and Gravemeijer 2006).

The term 'realistic' stresses that problem situations should be 'experientially real' for students (Cobb et al. 1992). This does not necessarily mean that the problem situations are always encountered in daily life. Students can experience an abstract mathematical problem as real when the mathematics of that problem is meaningful to them. Freudenthal's (1991) ideal was that mathematical learning should be an enhancement of common sense. Students should be allowed and encouraged to invent their own strategies and ideas, and they should learn mathematics on their own authority. At the same time, this process should lead to particular end goals. This process is called *guided reinvention*—one of the design heuristics of RME. This heuristic points to the question that underlies much of the RME-based research, namely that of how to support this process of engaging students in meaningful mathematical and statistical problem solving, and using students' contributions to reach certain end goals.

The theory of RME is especially tailored to mathematics education, because it includes specific tenets on and design heuristics for mathematics education. For a description of these tenets we refer to Treffers (1987) and for the design heuristics to Gravemeijer (1994) or Bakker and Gravemeijer (2006).

### *16.2.5 Methods*

The absence of the type of learning aimed for is a common reason to carry out design research. For Bakker's study in statistics education, descriptive, comparative, or evaluative research did not make sense because the type of learning aimed for could not be readily observed in classrooms. Considerable design and research effort first had to be taken to foster specific innovative types of learning. Bakker therefore had to design HLTs with accompanying educational materials that supported the desired type of learning about distribution. Design-based research offers a systematic approach to doing that while simultaneously developing domain-specific theories about how to support such learning for example here on the domain of statistics. In general, DBR researchers first need to create the conditions in which they can develop and test an instruction theory, but to create those conditions they also need research.

*Teaching experiment*. Bakker designed educational materials with accompanying HLTs in several cycles. Here we focus on the last cycle, involving a teaching experiment in grade 8. Half of the lessons were carried out in a computer lab and as part of them students used two minitools (Cobb et al. 1997), simple Java applets with which they analyzed data sets on, for instance, battery life span, car colours, and salaries (Fig. 16.3). The researcher was responsible for the educational materials and the teacher was responsible for the teaching, though we discussed in advance on a weekly basis both the materials and appropriate teaching style. Three preservice teachers served as assistants and helped with videotaping and interviewing students and with analyzing the data.

In the example that we elaborate we focus on the fourth of a series of ten lessons, each 50 min long. In this specific lesson, students reasoned about larger and larger samples and about the shape of distributions.

*Subjects.* The teaching experiment was carried out in an eighth-grade class with 30 students in a state school in the center of a Dutch city. The students in this study were being prepared for pre-university (*vwo*) or higher professional education (*havo*). The students in the class reported on here were not used to whole-class discussions, but rather to be "taken by the hand" as the teacher called it; they were characterized by the three research assistants as "passive but willing to cooperate." These students had no prior instruction in statistics; they were acquainted with bar and line graphs, but not with dot plots, histograms, or box plots. Students already knew the mean from calculating their report grades, but mode and median were not introduced until the second half of the educational sequence after variability, data, sampling, and shape had been topics of discussion.

*Data collection.* The collected data on which the results presented in this chapter are based include student work, field notes, and the audio and video recordings of class activities that the three assistants and researcher made in the classroom. An essential part of the data corpus was the set of mini-interviews we held during the lessons; they varied from about twenty seconds to four minutes, and were meant to find out what concepts and graphs meant for students, or how the minitools were used. These mini-interviews influenced students' learning because they often stimulated reflection. However, we think that the validity of the research was not put in danger by this, since the aim was to find out how students learned to reason with shape or distribution, not whether teaching the sequence in other eighth-grade classes would lead to the same results in the same number of lessons. Furthermore, the interview questions were planned in advance as part of the HLT, and discussed with the assistants.

*Retrospective analysis.* In this example we do not illustrate how HLTs can be compared with observed learning (see Dierdorp et al. 2011). Here we highlight one type of analysis that in Bakker's case yielded more theoretical insights: a method resembling Glaser and Strauss's constant comparative method (Glaser and Strauss 1967). For the analysis, Bakker watched the videotapes, read the transcripts, and formulated conjectures on students' learning on the basis of transcript episodes. Numbering the conjectures served as useful codes to work with during the analysis. Examples of such codes and conjectures were:

*C1*. Students divide imaginary data sets into three groups of low, 'average', and high values.

*C2*. Students either characterize spread as range or look very locally at spread

*C3*. Students are inclined to think of small samples when first asked about how one could test something (batteries, weight).

*C5*. What-if questions work well for letting students think of aggregate features of a graph or a situation. What would a weight graph of older students look like? What would the graph look like if a larger sample was taken? What would a larger sample of a good battery brand look like?

*C7*. Students' notions of spread, distribution, and density are not yet distinguished. When explaining how data are spread out, they often describe the distribution or the density in some area.

*C9*. Even when students see a large sample of a particular distribution, they often do not see the shape we see in it.

The generated conjectures were tested against other episodes and the rest of the collected data (student work, field observations, and tests) in the next round of anal-

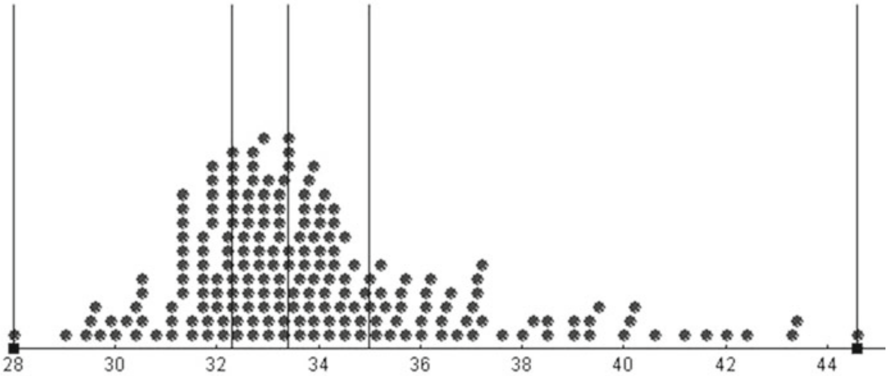



**Fig. 16.2** Jeans data with four equal groups option in Minitool 2

ysis by data triangulation. Conjectures that were confirmed remained in the list; conjectures that were refuted were removed from the list. Then the whole generating and testing process was repeated. The aforementioned examples were all confirmed throughout this analysis.

To get a sense of the interrater reliability of the analysis, about one quarter of the episodes including those discussed in this chapter and the conjectures belonging to these episodes were judged by the three assistants who attended the teaching experiment. The amount of agreement among judges was very high: all four judges agreed about 33 out of 35 codes. A code was only accepted if all judges agreed after discussion. We give an example of a code that was finally rejected and one that was accepted. This example stems from the seventh lesson in which two students used the four equal groups option in Minitool 2 for a revised version of the jeans activity. Their task was to advise a jeans factory about frequencies of jeans sizes to be produced (Fig. 16.2).

| | |
|---|---|
| Sofie | Because then you can best see the spread, how it is distributed. |
| Int. | How it is distributed. And how do you see that here [in this graph]? What do you look at then? (…) |
| Sofie | Well, you can see that, for example, if you put a [vertical] line here, here a line, and here a line. Then you see here [two lines at the right] that there is a very large spread in that part, so to speak. |

In the first line, Sofie seems to use the terms spread and distributed as almost synonymous. This line was therefore coded with C7, which states that "students' notions of spread, distribution, and density are not yet distinguished. When explaining how data are spread out, they often describe the distribution or the density in some area." In the second line, Sofie appears to look at spread very locally, hence it was coded with C2, which states that "students either characterize spread as range or look very locally at spread."

We also give an example of a code assignment that was dismissed in relation to the same diagram.

Int. What does this tell you? Four equal groups?
Melle Well, I think that most jeans are between 32 and 34 [inches].

We had originally assigned the code C1 to the this episode (students talk about data sets as consisting of three groups of low, 'average', and high values), because "most jeans are between 32 and 34" implies that below 32 and above 34 the frequencies are relatively low. In the episode, however, this student did not talk about three groups of low, average, and high values or anything equivalent. We therefore removed the code from this episode.

### *16.2.6 HLT and Retrospective Analysis*

To illustrate relationships between theory, method, and results, this section presents the analysis of students' reasoning during one educational activity which was carried out in the fourth lesson. Its goal was to stimulate students to reason about larger and larger samples. We summarize the HLT of that lesson: the learning goal, the activity of growing a sample and the assumptions about students' potential learning processes and about how the teacher could support these processes. We then present the retrospective analysis of three successive phases in growing a sample.

The overall *goal* of the growing samples activity as formulated in the hypothetical learning trajectory for this fourth lesson was to stimulate students' diagrammatic reasononing about shape in relation to sampling and distribution aspects in the context of weight. This implied that students should first make diagrams, then experiment with them and reflect on them. The idea was to start with ideas invented by the students and guide them toward more conventional notions and representations. This process of guiding students toward these culturally accepted concepts and graphs while building on their own inventions is called guided reinvention. We had noted in previous teaching experiments that students were inclined to choose very small samples initially. It proved necessary to stimulate reflection on the disadvantages of such small samples and have them predict what larger samples would look like. Such insights from the analyses of previous teaching experiments helped to better formulate the HLT of a new teaching experiment. More particularly, Bakker assumed that starting with students' initial ideas about small samples and asking for predictions about larger samples would make students aware of various features of distributions.

The *activity* of growing a sample consisted of three phases of making sketches of a hypothetical situation and comparing those sketches with graphs displaying real data sets. In the first phase students had to make a graph of their own choice of a predicted weight data set with sample size 10. The results were discussed by the teacher to challenge this small sample size, and in the subsequent phases students had to predict larger data sets, one class and three classes in the second phase, and all students in the province in the third phase. Thus, three such phases took place as

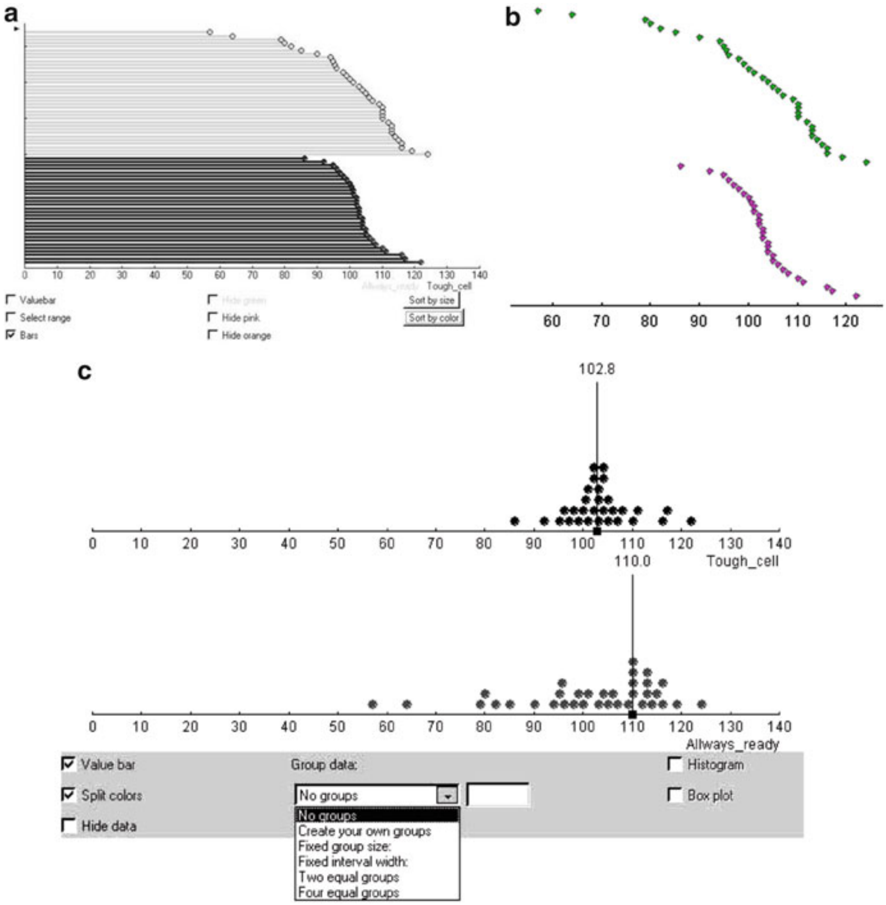



**Fig. 16.3** (**a**) Minitool 1 showing a value-bar graph of battery life spans in hours of two brands. (**b**) Minitool 1, but with bars hidden. (**c**) Minitool 2 showing a dot plot of the same data sets

described and analyzed below. Aiming for guided reinvention, the teacher and researcher tried to strike a balance between engaging students in statistical reasoning and allowing their own terminology on the one hand, and guiding them in using conventional and more precise notions and graphical representations on the other. Figure 16.3b is the result of focusing only on the endpoints of the value bars in Fig. 16.3a. Figure 16.3c is the result of these endpoints falling down vertically on the x-axis. In this way, students can learn to understand the relationship between value-bar graphs and dot plots, and what distribution features in different representations look like (Bakker and Hoffmann 2005).

#### 16.2.6.1 Analysis of the First Phase of Growing a Sample

The text of the student activity sheet for the fourth lesson contained a number of tasks that we cite in the following subsections. The sheet started as follows:

> Last week you made graphs of predicted data for a balloon pilot. During this lesson you will get to see real weight data of students from another school. We are going to investigate the influence of the sample size on the shape of the graph.
>
> Task a. Predict a graph of ten data values, for example with the dots of minitool 2.

The sample size of ten was chosen because the students had found that size reasonable after the first lesson in the context of testing the life span of batteries. Figure 16.4 shows examples for three different types of diagrams the students made to show their predictions: there were three value-bar graphs (such as in minitool 1—e.g., Ruud's diagram), eight with only the endpoints (such as with the option of minitool 1 to "hide bars"—e.g., Chris's diagram) and the remaining nineteen plots were dot plots (such as in minitool 2—e.g., Sandra's diagram). For the remainder of this section, the figures and written explanations of these three students are demonstrated, because their work gives an impression of the variety of the whole class. Those three students were chosen because their diagrams represent all types of diagrams made in this class, also for other phases of growing a sample.

To stimulate the reflection on the graphs, the teacher showed three samples of ten data points on the blackboard and students had to compare their own graphs (Fig. 16.4) with the graphs of the real data sets (Fig. 16.5).

> Task b. You get to see three different samples of size 10. Are they different from your own prediction? Describe the differences.

The reason for showing three small samples was to show the variation among these samples. There were no clear indications, though, that students conceived this variation as a sign that the sample size was too small for drawing conclusions, but they generally agreed that larger samples were more reliable. The point relevant to the analysis is that students started using predicates to describe aggregate features of the graphs. The written answers of the three students were the following:

| | |
|---|---|
| Ruud | Mine looks very much like what is on the blackboard. |
| Chris | The middle-most [diagram on the blackboard] best resembles mine because the weights are close together and that is also the case in my graph. It lies between 35 and 75 [kg]. |
| Sandra | The other [real data] are more weights together and mine are further apart. |

Ruud's answer is not very specific, like most of the written answers in the first phase of growing samples. Chris used the predicate "close together" and added numbers to indicate the range, probably as an indication of spread. Sandra used such terms as "together" and "further apart," which address spread. The students in the class used common predicates such as "together," "spread out" and "further apart" to describe features of the data set or the graph. For the analysis it is important to

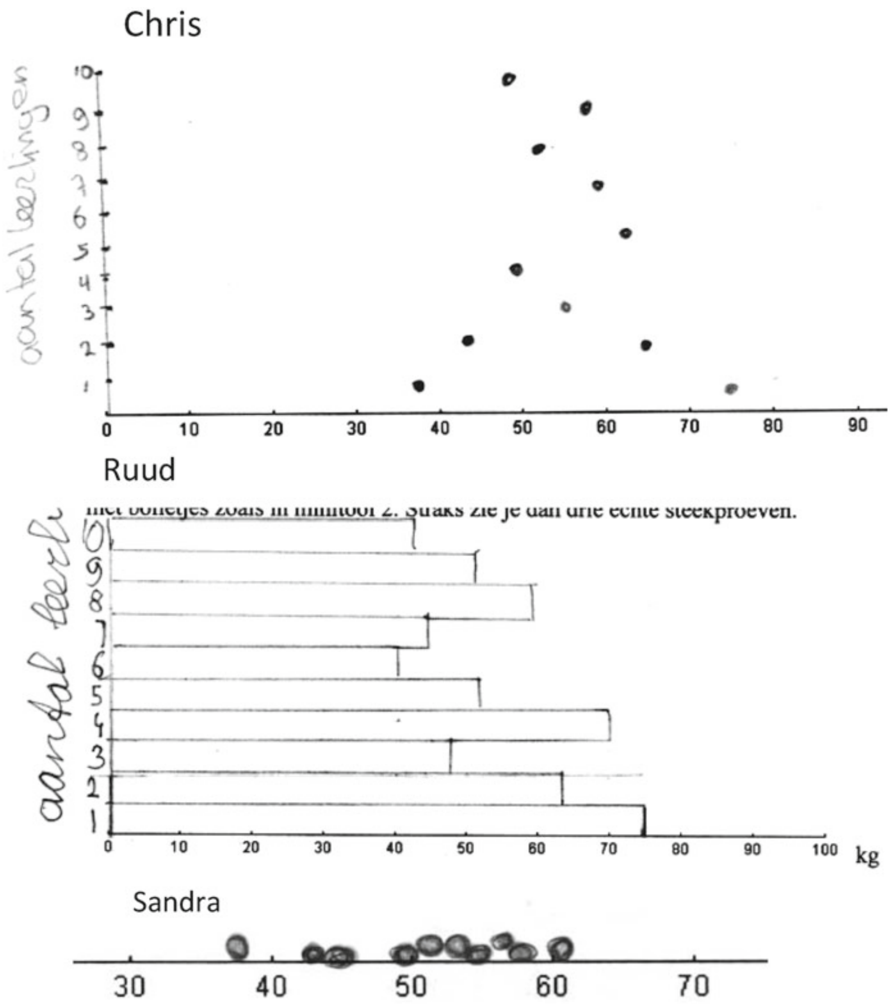
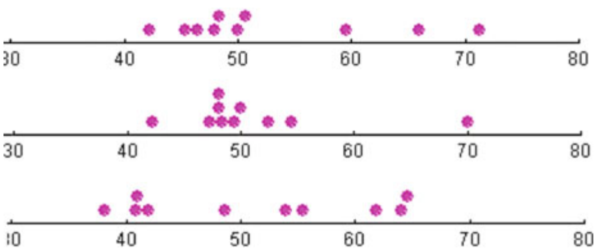



**Fig. 16.4** Student predictions (Ruud, Chris, and Sandra) for ten data points (weight in kg) (Bakker 2004a, p. 219)

**Fig. 16.5** Three real data sets in minitool 2 (Bakker 2004a, p. 219)



note that the students used predicates (together, apart) and no nouns (spread, average) in this first phase of growing samples. Spread can only become an object-like concept, something that can be talked about and reasoned with, if it is a noun. In the semiotic theory of Peirce, such transitions from the predicate "the dots are spread out" to "the spread is large" are important steps in the formation of concepts (see Bakker and Derry 2011, for our view on concept formation).

#### 16.2.6.2 Analysis of the Second Phase of Growing a Sample

The students generally understood that larger samples would be more reliable. With the feedback students had received after discussing the samples of ten data points in dot plots, students had to predict the weight graph of a whole class of 27 students and of three classes with 67 students (27 and 67 were the sample sizes of the real data sets of eighth graders of another school).

> Task c. We will now have a look how the graph changes with larger samples. Predict a sample of 27 students (one class) and of 67 students (three classes).
>
> Task d. You now get to see real samples of those sizes. Describe the differences. You can use words such as majority, outliers, spread, average.

During this second phase, all of the students made dot plots, probably because the teacher had shown dot plots on the blackboard, and because dot plots are less laborious to draw than value bars (only one student started with a value-bar graph for the sample of 27, but switched to a dot plot for the sample of 67). The hint on statistical terms was added to make sure that students' answers would not be too superficial as (often happened before) and to stimulate them to use such notions in their reasoning. It was also important for the research to know what these terms meant to them. When the teacher showed the two graphs with real data, once again there was a short class discussion in which the teacher capitalized on the question of why most student predictions now looked pretty much like what was on the blackboard, whereas with the earlier predictions there was much more variation. No student had a reasonable explanation, which indicates that this was an advanced question. The figures of the same three students are presented in Figs. 16.6 and 16.7 and their written explanations were:

| | |
|---|---|
| Ruud | My spread is different. |
| Chris | Mine resembles the sample, but I have more people around a certain weight and I do not really have outliers, because I have 10 about the 70 and 80 and the real sample has only 6 around the 70 and 80. |
| Sandra | With the 27 there are outliers and there is spread; with the 67 there are more together and more around the average. |

Here, Ruud addressed the issue of spread ("my spread is different"). Chris was more explicit about a particular area in her graph, the category of high values. She also correctly used the term "sample," which was newly introduced in the second lesson. Sandra used the term "outliers" at this stage, by which students meant "extreme values," which did not necessarily mean exceptional or suspect values.

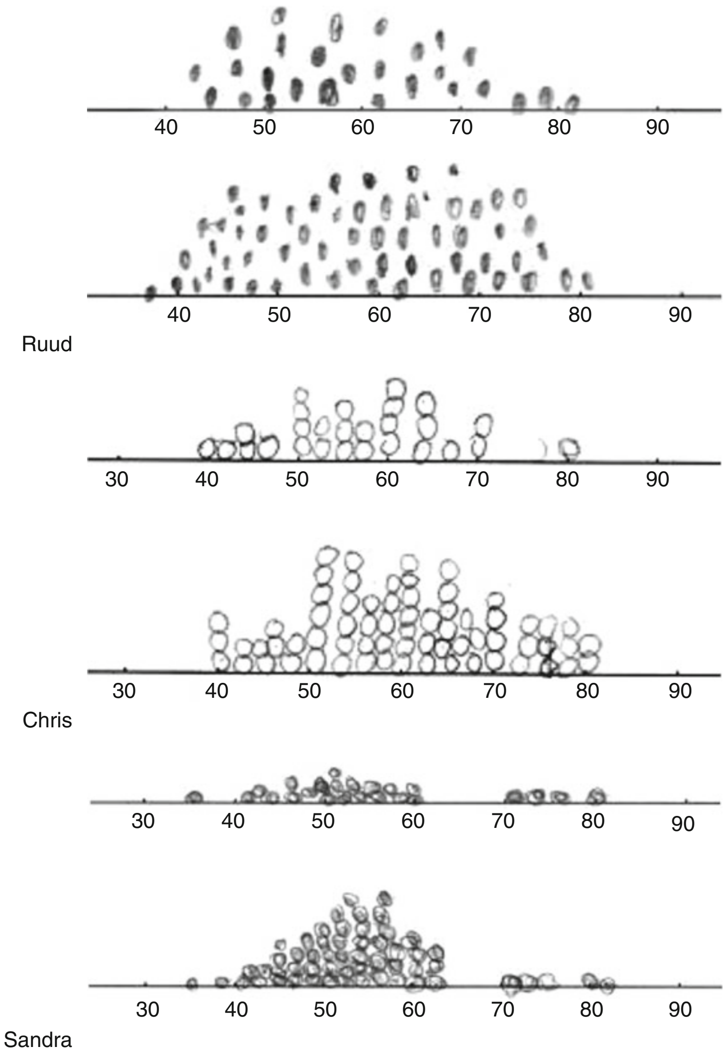



**Fig. 16.6** Predicted graphs for one class (n = 27, top plot) and three classes (n = 67, bottom plot) by Ruud, Chris, and Sandra (Bakker 2004a, p. 222)

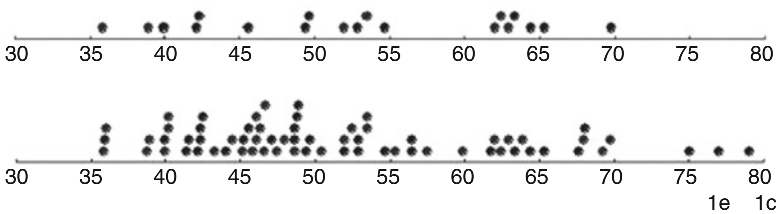



**Fig. 16.7** Real data sets of size 27 and 67 of students from another school (Bakker 2004a, p. 222)

She also seemed to locate the average somewhere and to understand that many students are about average. These examples illustrate that students used statistical notions for describing properties of the data and diagrams.

In contrast to the first phase of growing a sample, students used nouns instead of just predicates for comparing the diagrams. Like others Ruud used the noun “spread” (“my spread is different”) whereas students earlier used only predicates such as “spread out” or “further apart” (e.g., Sandra). Of course, this does not always imply that if students use these nouns that they are thinking of the right concept. Statistically, however, it makes a difference whether we say, “the dots are spread out” or “the spread is large.” In the latter case, spread is an object-like entity that can have particular aggregate characteristics that can be measured, for instance by the range, the interquartile range, or the standard deviation. Other notions such as outliers, sample, and average, are now used as nouns, that is as conceptual objects that can be talked about and reasoned with.

#### 16.2.6.3 Analysis of the Third Phase of Growing a Sample

The aim of the hypothetical learning trajectory was that students would come to draw continuous shapes and reason about them using statistical terms. During teaching experiments in the seventh-grade experiments (Bakker and Gravemeijer 2004), reasoning with continuous shapes turned out to be difficult to accomplish, even if it was asked for. It often seemed impossible to nudge students toward drawing the general, continuous shape of data sets represented in dot plots. At best, students drew spiky lines just above the dots. This underlines that students have to construct something new (a notion of signal, shape, or distribution) with which they can look differently at the data or the variable phenomenon.

In this last phase of growing the sample, the task was to make a graph showing data of all students in the city, not necessarily with dots. The intention of asking this was to stimulate students to use continuous shapes and dynamically relate samples to populations, without making this distinction between sample and population explicit yet. The conjecture was that this transition from a discrete plurality of data values to a continuous entity of a distribution is important to foster a notion of distribution as an object-like entity with which students could model data and describe aggregate properties of data sets. The task proceeded as follows:

> Task e. Make a weight graph of a sample of all eighth graders in the city. You need not draw dots. It is the shape of the graph that is important.
>
> Task f. Describe the shape of your graph and explain why you have drawn that shape.

The figures of the same three students are presented in Fig. 16.8 and their written explanations were:

Ruud Because the average [values are] roughly between 50 and 60 kg.
Chris I think it is a pyramid shape. I have drawn my graph like that because I found it easy to make and easy to read.
Sandra Because most are around the average and there are outliers at 30 and 80 [kg].

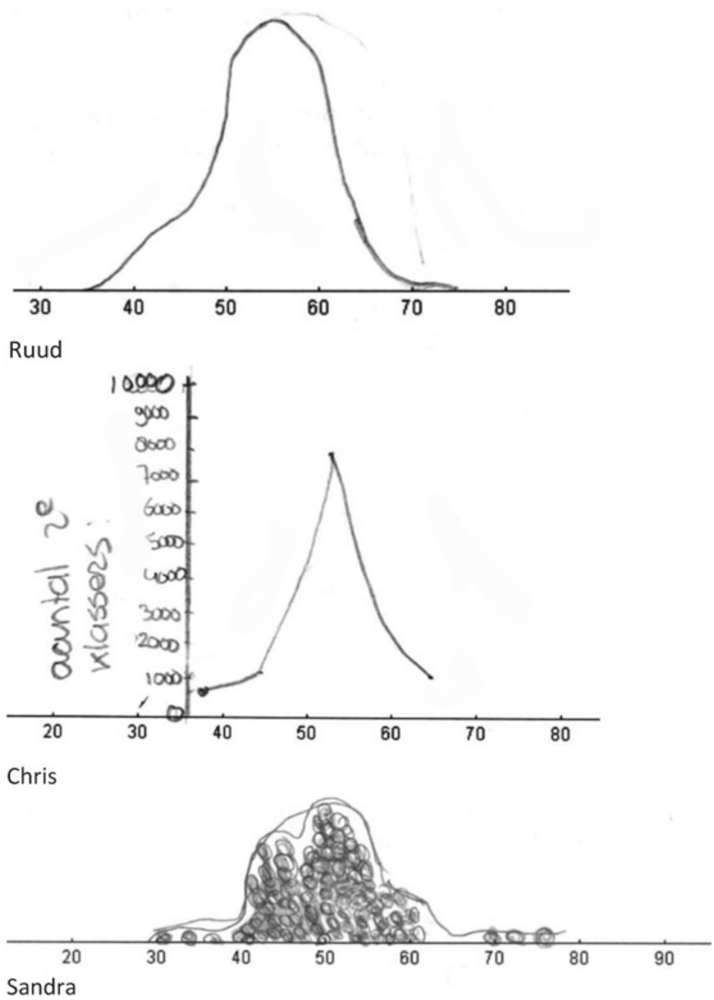



**Fig. 16.8** Predicted graphs for all students in the city by Ruud, Chris, and Sandra (Bakker 2004a, p. 224)

Ruud's answer focused on the average group. During an interview after the fourth lesson, Ruud like three other students literally called his graph a "bell shape," though he had probably not encountered that term in a school situation before. This is probably a case of *reinvention*. Chris's graph was probably inspired by line graphs that the students made during mathematics lessons. She introduced the vertical axis with frequency, though such graphs had not been used before in the statistics course. Sandra may have started with the dots and then drawn the continuous shape.

In this third phase of growing a sample, 23 students drew a bump shape. The words they used for the shapes were pyramid (three students), semicircle (one), and bell shape (four). Many students drew continuous shapes but these were all symmetrical. Since weight distributions are not symmetrical and because skewness is an important concept, a subsequent lesson addressed asymmetrical shapes in relation to the weight data (see Bakker 2004b).

### *16.2.7 Reflection on the Example*

The research question we addressed in the example is: How can coherent reasoning about distribution be promoted in relation to data, variability, and sampling in a way that is meaningful for students with little statistical background? We now discuss those key elements for the educational activity and speculate about what can be learned from the analysis presented here.

The activity of growing a sample involved short phases of constructing diagrams of new hypothetical situations, and comparing these with other diagrams of a real sample of the same size. The activity has a broader empirical basis than just the teaching experiment reported in this chapter, because it emerged from a previous teaching experiment (Bakker and Gravemeijer 2004) as a way to address shape as a pattern in variability.

To theoretically generalize the results, Bakker analyzed students' reasoning as an instance of diagrammatic reasoning, which typically involves constructing diagrams, experimenting with them, and reflecting on the results of the previous two steps. In this growing samples activity, the quick alternation between prediction and reflection during diagrammatic reasoning appears to create ample opportunities for concept formation, for instance of spread.

In the first phase involving the prediction of a small data set, students noted that the data were more spread out, but in subsequent phases, students wrote or said that the spread was large. From the terms used in this fourth lesson, we conclude that many statistical concepts such as center (average, majority), spread (range and range of subsets of data), and shape had become topics of discussion (object-like entities) during the growing samples activity. Some of these words were used in a rather unconventional way, which implies that students needed more guidance at this point. Shape became a topic of discussion as students predicted that the shape of the graph would be a semicircle, a pyramid, or a bell shape, and this was exactly what the HLT targeted. Given the students' minimal background in statistics and the fact that this was only the fourth lesson of the sequence, the results were promising. Note, however, that such activities cannot simply be repeated in other contexts; they need to be adjusted to local circumstances if they are to be applied in other situations.

The instructional activity of growing samples later became a connecting thread in Ben-Zvi's research in Israel, where it also worked to help students develop statistical concepts in relation to each other (Ben-Zvi et al. 2012). This implies that this instructional idea was transferable to other contexts. The transferability of instructional ideas from the USA to the Netherlands to Israel, even to higher levels of education, illustrates that generalization in DBR can take place across contexts, cultures and age group.

### 16.2.8 Final Remarks

The example presented in Sect. 16.2 was intended to substantiate the issues discussed in Sect. 16.1, and we hope that readers will have a sense of what DBR could look like and feel invited to read more about it. It should be noted that there are many variants of DBR. Some are more focused on theory, some more on empirically grounded products. Some start with predetermined learning outcomes, others have more open-ended goals (cf. Engeström 2011). DBR may be a challenging research approach but it is in our experience also a very rewarding one given the products and insights that can be gained.

**Acknowledgments** The research was funded by the Netherlands Organization for Scientific Research under grant number 575-36-003B. The writing of this chapter was made possible with a grant from the Educational and Learning Sciences Utrecht awarded to Arthur Bakker. Section 2.6 is based on Bakker (2004b). We thank our Master students in our Research Methodology courses for their feedback on earlier versions of this manuscript. Angelika Bikner-Ahsbahs's and reviewers' careful reading has also helped us tremendously. We also acknowledge PhD students Adri Dierdorp, Al Jupri, and Victor Antwi, and our colleague Frans van Galen for their helpful comments, and Nathalie Kuijpers and Norma Presmeg for correcting this manuscript.

## Appendix: Structure of a DBR Project with Illustrations

In line with Oost and Markenhof (2010), we formulate the following general criteria for any research project:

1. The research should be **anchored** in the literature.
2. The research aim should be **relevant**, both in theoretical and practical terms.
3. The formulation of aim and questions should be **precise**, i.e. using concepts and definitions in the correct way.
4. The method used should be **functional** in answering the research question(s).
5. The overall structure of the research project should be **consistent**, i.e. title, aim, theory, question, method and results should form a coherent chain of reasoning.

In this appendix we present a structure of general points of attention during DBR and specifications for our statistics education example, including references to relevant sections in the chapter. In this structure these criteria are bolded. This structure could function as the blueprint of a book or article on a DBR project.

| | General points | Examples |
|---|---|---|
| Introduction: | 1. Choose a topic | 1. Statistics education at the middle school level |
| | 2. Identify common problems | 2. Statistics as a set of unrelated concepts and techniques |
| | 3. Identify knowledge gap and relevance | 3. How middle school students can be supported to develop a concept of distribution and related statistical concepts |
| | 4. Choose mathematical learning goals | 4. Understanding of distribution (2.1) |

(continued)

| | General points | Examples |
|---|---|---|
| Literature review forms the basis for formulating the research aim (the research has to be **anchored** and **relevant**) | | |
| Research aim: | It has to be clear whether an aim is descriptive, explanatory, evaluative, advisory etc. (1.2.2) | Contribute to an empirically and theoretically grounded instruction theory for statistics education at the middle school level (advisory aim) (2.1) |
| Research aim has to be narrowed down to a research question and possibly subquestions with the help of different theories | | |
| Literature review (theoretical background): | Orienting frameworks | Semiotics (2.3) |
| | Frameworks for action | Theories on learning with computer tools |
| | Domain-specific learning theories (1.2.8) | Realistic Mathematics Education (2.4) |
| With the help of theoretical constructs the research question(s) can be formulated (the formulation has to be **precise**) | | |
| Research question: | Zoom in what knowledge is required to achieve the research aim | How can students with little statistical background develop a notion of distribution? |
| It should be underpinned why this research question requires DBR (the method should be **functional**) | | |
| Research approach: | The lack of the type of learning aimed for is a common reason to carry out DBR: It has to be enacted so it can be studied | Dutch statistics education was atomistic: Textbooks addressed mean, median, mode, and different graphical representations one by one. Software was hardly used. Hence the type of learning aimed for had to be enacted. |
| Using a research method involves several research instruments and techniques | | |
| Research instruments and techniques | Research instrument that connects different theories and concrete experiences in the form of testable hypotheses. | Series of hypothetical learning trajectories (HLTs) |
| | 1. Identify students' prior knowledge | 1. Prior interviews and pretest |
| | 2. Professional development of teacher | 2. Preparatory meetings with teacher |
| | 3. Interview schemes and planning | 3. Mini-interviews, observation scheme |
| | 4. Intermediate feedback and reflection with teacher | 4. Debrief sessions with teacher |
| | 5. Determine learning yield (1.4.2) | 5. Posttest |
| Design | Design guidelines | Guided reinvention; Historical and didactical phenomenology (2.4) |
| Data analysis | Hypotheses have to be tested by comparison of hypothetical and observed learning. Additional analyses may be necessary (1.4.3) | Comparison of hypothetical and observed learning |
| | | Constant comparative method of generating conjectures and testing them on the remaining data sources (2.6) |

(continued)

| | General points | Examples |
|---|---|---|
| Results | Insights into patterns in learning and means of supporting such learning | Series of HLTs as progressive diagrammatic reasoning about growing samples (2.6) |
| Discussion | Theoretical and practical yield | Concrete example of an historical and didactical phenomenology in statistics education |
| | | Application of semiotics in an educational domain |
| | | Insights into computer use in the mathematics classroom |
| | | Series of learning activities |
| | | Improved computer tools |
| The aim, theory, question, method and results should be aligned (the research has to be **consistent**) | | |

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# Chapter 17
# Perspectives on Design Research: The Case of Didactical Engineering

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**Abstract** In what is often called the "French didactical culture," design has always played an essential role in research. This is attested by the introduction and institutionalization of a specific concept, that of *didactical engineering*, already in the early 1980s and by the way didactical engineering has accompanied the development of didactical research, both in its fundamental and applied dimensions. In this chapter, I present this vision of design and its characteristics as a research methodology, coming back to its historical origin in close connection with the development of the theory of didactical situations, tracing its evolution along the last three decades, and illustrating this methodology by some particular examples. I also consider current developments within this design culture, especially those linked to the integration of a design dimension into the anthropological theory of didactics and also to the idea of didactical engineering of second generation introduced for addressing more efficiently the development dimension of didactical engineering.

**Keywords** Didactical engineering • Theory of didactical situations

## 17.1 Introduction

Design has always played a substantial role in mathematics education up to the point that some researchers consider this field as a design science (see, for instance, Wittmann 1998; Cobb 2007). But the conception of design and the exact role it is given in research strongly depend on educational cultures. In this chapter we consider the case of what is often called the "French didactical culture" in which design has always played a fundamental role. This importance of design is attested by the introduction and institutionalization of a specific concept, that of *Didactical*

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A. Bikner-Ahsbahs et al. (eds.), *Approaches to Qualitative Research in Mathematics Education*, Advances in Mathematics Education,
DOI 10.1007/978-94-017-9181-6_17

*Engineering* (DE in the following) already in the early eighties. Since that time DE, which developed in close connection with the theory of didactical situations initiated by Brousseau (cf. (Warfield 2006) for an introduction and (Brousseau 1997) for a more detailed vision), has accompanied the development of didactical research, both in its fundamental and applied dimensions. This chapter is structured into four main sections. In the first section I briefly review the development of DE from its emergence in the early eighties until now, and clarify its links with the theory of didactical situations (see also (Bessot 2011)). In the second section I present its characteristics as a research methodology. In the third section I illustrate this methodology with examples taken at different levels of schooling. In the fourth section I consider two recent evolutions of DE. The first one is conveyed by the anthropological theory of didactics in terms of course of study and research that considers very open forms of design; the second one is "didactical engineering of second generation" introduced by Perrin-Glorian for addressing dissemination and up-scaling issues (Perrin-Glorian 2011). Beyond the many examples of realizations, the writing of this chapter has been especially inspired by some foundational texts such as (Chevallard 1982; Artigue 1990, 2002, 2009), and by the extensive reflection on didactical engineering carried out at the XV^e Summer School of Didactics of Mathematics in 2009 (Margolinas et al. 2011).

## 17.2 Didactical Engineering: An Historical Review

The emergence, consolidation and evolution of didactical engineering can be traced through the successive summer schools of didactics of mathematics organized every 2 years in France since 1980. In this brief historical review, I focus on three of these (1982, 1989, 2009) for which DE was a specific theme of study. Already, at the second summer school in 1982, DE was one of the themes addressed. Chevallard prepared a specific manuscript note for supporting the work of the summer school collective (Chevallard 1982); Brousseau gave a course, and practical sessions were organized around this theme. Accessible documents regarding this summer school show the shared conviction that didactical research should give a more central role to the construction and study of classroom realizations. French researchers expressed concerns about the observed tendency to privilege methodologies borrowed from established fields such as psychology (clinical interviews, questionnaires, pre-test/post-test comparisons…) for ensuring the scientific legitimacy of research in mathematics education. They pointed out that the didactics of mathematics is a genuine scientific field whose methodologies should be in line with its specific purpose: the study of intentional dissemination of mathematical knowledge through didactical systems, and the associated interaction between teaching and learning processes. As explained in Chevallard's note, the need for developing specific methodologies based on classroom realizations was justified by both theoretical and practical reasons. On the theoretical side, such methodologies were judged necessary for this essential part of scientific activity which is the production of phenomena (in this case, didactical phenomena), what Bachelard (1937) called *phénoménotechnique*. On the practical side, such methodologies were judged necessary for establishing productive relationships between research and practice, because they permit researchers to consider didactical systems in their concrete functioning, and to pay the necessary attention to the different constraints and forces acting on these, which could be neglected otherwise. Didactical engineering thus emerged as a research and development methodology based on classroom realizations in form of sequences of lessons, informed by theory and putting to the test theoretical ideas. At that time, what was predominant in the French didactical community was the theory of didactical situations that had emerged in the late 1960s. This theory became thus the natural support of DE. Its systemic perspective, constructions and values shaped DE, which progressively became the research methodology privileged within this community. In fact, it would be more adequate to say that theoretical constructions and DE jointly developed along the 1980s.

In 1989, for the second time, didactical engineering was a specific theme of the summer school and I was asked to give a course on this methodology. This course (Artigue 1990) contributed to the institutionalization of DE as a research methodology, making explicit its characteristics and its foundational links with the theory of didactical situations. It also pointed out that its privileged links with the theory of didactical situations did not prevent researchers using this methodology from relying on other theoretical approaches. For instance, several examples mentioned in the course or worked out in the practical sessions associated to it relied on the tool-object dialectics due to Douady (1986). Many contributions to the summer school indeed combined its specific constructs (through the attention paid in design to the dialectics to be organized between the tool and object dimensions of mathematics concepts and to the learning potential offered by moves between mathematical settings, numbers and geometry for instance) with those offered by the theory of didactical situations. In this course too, I pointed out that if DE had consolidated as a research methodology, the problem of establishing productive links between research and practice had not been solved. DE produced by research was disseminating through articles, educational resources and teacher education, but there was some evidence that along this dissemination process, it tended to lose its essence and value.

In fact, in coherence with the theory of didactical situations, in DE design, particular efforts had been made to create situations in which:

- the mathematical knowledge aimed at is an optimal solution to the problem to solve (which is captured in the theory by the idea of *fundamental situation*);
- students as a collective are able to reach this optimal solution through their interaction with the *milieu*[1] of the situation, without significant help from their teacher (which is captured in the theory by the idea of *adidactical situation*).

[1] In the theory, the milieu of a situation is defined as the system with which the student interacts, and which provides objective feedback to her. The milieu may comprise material and symbolic elements: artifacts, informative texts, data, results already obtained…, and also other students who collaborate or compete with the learner.

The teacher's role, for its part, had been mainly approached in terms of the dual processes of *devolution* and *institutionalization*, coherently with the vision of learning as a combination of *adaptation* and *acculturation* processes underlying the theory. Through the devolution process, the teacher tries to make her students accept the mathematical responsibility of solving the problem at stake. She tries to make thus possible the adidactic interaction with the milieu required for learning through adaptation. If the devolution process is successful, the students agree to forget for a while the didactical intention of the teacher; to concentrate on the search for mathematical solutions instead of trying to decipher the teacher's expectations. Through the process of institutionalization, the teacher connects the knowledge built by students through adidactic interaction with the milieu to the scholarly and decontextualized forms of knowledge aimed at by the institution, making the acculturation dimension of learning possible.

In 1989, even if the DEs produced by researchers had been able to approach in many cases this ideal-type, their functioning out of the control of research seemed difficult. Moreover, high attention was paid to the innovative situations designed for introducing new mathematical ideas or overcoming *epistemological obstacles,*[2] and much less to the more standard situations used for consolidating mathematical knowledge and techniques. This situation created a distorted vision of DE products that certainly had negative impact on the quality of their dissemination.

In 2009, 20 years later, DE was once again a theme for the summer school, in fact its unique theme. Since 1989, the didactic field had substantially evolved. The anthropological theory of didactics that was just emerging in the late 1980s had matured and gained in influence. Moreover, in the last decade, it had created its own design approach in terms of activities of research and study and then programs of study and research (Chevallard 2006, in press). A new theoretical framework had also emerged from the theory of didactical situations and the anthropological theory of didactics: the theory of joint action between teachers and students, proposing a renewed vision of the role of the teacher and of students-teacher interactions (Sensevy 2011, 2012). More generally, teachers' practices and professional development had become a focus of research, and this research had developed its own methodologies involving naturalistic and participative observations of classrooms. DE was still an important research methodology, especially each time the didactical systems one wanted to study could not be observed in natural conditions (as is for instance often the case in research about technology), but was no longer the privileged methodology (Artigue 2002, 2009). Didactical engineering had also migrated outside its original habitat. It has been extended to teacher education and to the study

[2] The notion of epistemological obstacle, introduced by the philosopher Gaston Bachelard, was imported in the educational field by Guy Brousseau (1983) for expressing the fact that the development of mathematical knowledge necessarily faces obstacles, due to prior forms of knowledge that were relevant and successful in specific contexts. Epistemological obstacles are those attested in the historical development of knowledge, and having played a constitutive role in this development. Their identification may help understand students' resistant errors and difficulties. Schneider (2014) provides a synthetic presentation and discussion of the notion, its development and use in mathematics education research.

of innovative pedagogical practices, including informal education; didacticians from other disciplines, for instance physical sciences or sports, had used it (Terrisse 2002); researchers educated in other countries and cultures, and having different theoretical backgrounds, had used it, for instance researchers referring to the socio-epistemological framework in mathematics education (Farfán 1997; Cantoral and Farfán 2003) or to semiotic approaches (Maschietto 2002; Falcade 2006). Moreover, design-based research perspectives had emerged and grown in other contexts, independently of it (Burkhardt and Schoenfeld 2003; Design-Based Research Collaborative 2003). These conditions created the need for a thorough reflection about the concept of DE and this was the exact purpose of the 2009 summer school. I have integrated some of the results of this reflection in the next section describing the characteristics of DE as a research methodology, and some others will be dealt with in the fourth section. Nevertheless the size of this chapter does not allow to pay full justice to the work carried out at this summer school and those interested are invited to read the report by Margolinas et al. (2011).

## 17.3 Didactical Engineering as a Research Methodology

In this section, I present the characteristics of DE as a research methodology, using for that purpose its most standard form: the conception, realization, observation, analysis and evaluation of classroom realizations aiming at the learning of a specific content. However, it should be clear that, while obeying fixed principles, this research methodology might take a diversity of forms in practice, according to the nature of the questions addressed by the researchers, and to the contexts involved. I will end this section by pointing out some similarities and differences with design-based research perspectives more and more influential in mathematics education.

One essential characteristic of DE as a research methodology is that, contrary to the traditional use of classroom realizations in educational research, it does not obey the validation paradigm based on the comparison of control and experimental groups. Its validation is internal and based on the comparison between the *a priori* and *a posteriori analyses* of the didactic situations involved. This methodological choice can be easily understood considering the educational culture in which DE has emerged. In this culture, as explained above, research in mathematics education (didactics of mathematics) is seen as a scientific field of its own whose ambition is the study of the intentional dissemination of mathematical knowledge through didactical systems. What is to be understood is the functioning of such didactical systems, and associated didactical phenomena, which requires entering into the intimacy of their functioning. Validating the hypotheses engaged in the conception phase of a DE cannot be thus a matter of comparison between experimental and control groups.

As a research methodology, DE is structured into different phases. These are the following: preliminary analyses, conception and *a priori* analysis, realization, observation and data collection, *a posteriori* analysis and validation.

### *17.3.1 Preliminary Analyses*

Preliminary analyses set the background for the conception phase of the process. They cover different dimensions, and especially the three following:

- An epistemological analysis of the content at stake, often including an historical part. This analysis helps researchers to fix the precise goals of the DE and to identify possible epistemological obstacles to be faced. It also supports the search for mathematical situations representative of the knowledge aimed at, what the theory of didactical situations calls *fundamental situations*. These are problematic situations for the solving of which this knowledge is necessary or in some sense optimal. The epistemological analysis helps the researchers to take the necessary reflective position and distance with respect to the educational world they are embedded in, and to build a reference point.
- An institutional analysis whose aim is to identify the characteristics of the context in which the DE takes place, the conditions and constraints it faces. These conditions and constraints may be situated at different levels of what is called the *hierarchy of levels of co-determination* (Chevallard 2002) in the anthropological theory of didactics. They may be attached to curricular choices regarding the content at stake and associated teaching practices, to more general curricular characteristics regarding the teaching of the discipline, the (technological) resources accessible, the evaluation practices and the school organization. They can also be linked to the characteristics of the students and teachers involved, to the way the school is connected with its environment… Depending on the precise goals and context of the research, the importance attached to these different levels may of course vary.
- A didactical analysis whose aim is to survey what research has to offer regarding the teaching and learning of the content at stake, and is likely to guide the design.

The three dimensions organizing the phase of preliminary analyses reflect the systemic perspective underlying DE as a research methodology. Each of them has its methodological specificities and needs. The epistemological analysis often involves the use of historical sources and not just secondary sources; the institutional analysis also generally includes an historical dimension. As made clear by the theory of didactical transposition (Bosch and Gascón 2006), curricular organizations and choices are the result of a long-term historical process; they cannot be understood just by analyzing current curricula, official documents and textbooks. Such understanding is needed for making clear the strength of the constraints faced and the way some of these can be moved in the design. The didactical analysis has generally a substantial cognitive dimension, but this cognitive dimension is only one part of the global picture even if what is aimed at is the development of a didactical strategy allowing students to learn better some part of mathematics.

It must also be pointed out that, according to the precise goals of the research, what is exactly investigated in these dimensions, and the respective importance attached to each of them may vary substantially.

### 17.3.2 Conception and a Priori Analysis

Conception and *a priori* analysis is a crucial phase of the methodology. It relies on the preliminary analyses carried out, and is the place where research hypotheses are made explicit and engaged in the conception of didactical situations, where theoretical constructs are put to the test. Conception requires a number of choices and these situate at different levels. Some choices pilot the global project and in that case it is usual to speak of *macro-choices;* some are situated at the level of a particular situation, and in that case it is usual to speak of *micro-choices*. These choices determine *didactical variables,*[3] so we have both *macro-didactic and micro-didactic variables*. These variables condition the milieu, thus the interactions between students and knowledge, the interactions between students and between students and teacher, thus the exact opportunities that students have to learn, how and what they can learn. In line with the theoretical foundations of DE, in these choices particular attention must be paid to the epistemological pertinence of the problems posed and to the mathematical responsibility given to the students.

The *a priori* analysis makes clear the different choices and the way they relate to the research hypotheses and preliminary analyses. For each situation, it identifies the main didactical variables, that is to say those that affect the efficiency and cost of the possible strategies developed by students, and their possible dynamics. These variables can be attached to the characteristics of the tasks proposed to students, but they can also be linked to the resources provided to the students for solving these tasks (which in the theory corresponds to the *material milieu* of the situation) and to the way the students' interaction with the *milieu* is socially organized. From these characteristics, conjectures are made regarding the possible development of the situation, students' interaction with the *milieu*, students' strategies and their evolution, and the possible sharing of mathematical responsibilities between teacher and students. It is important to stress that such conjectures do not regard individuals but a *generic and epistemic student* who enters the situation with some supposed knowledge background and is ready to play the role that the situation proposes her to play. Of course, the realization will involve students with their personal specificities and history, but the goal of the *a priori* analysis is not to anticipate how each particular student will behave and benefit from the situation, but what the situation *a priori* can offer in terms of learning in the context at stake. It creates a reference with which classroom realizations will be contrasted.

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[3]Among the many variables influencing the possible dynamics of a situation and its learning outcomes, didactical variables are those under the control of the teacher. In a situation of enlargement such as the well-known "Puzzle situation" by Brousseau, the number of pieces of the puzzle, their shapes and dimensions, the ratio of enlargement are didactical variables; the fact that students work in group, each student being asked to enlarge one piece of the puzzle is also a didactical variable.

### 17.3.3 Realization, Observation and Data Collection

During the realization phase, data are collected for the analysis *a posteriori*. The nature of the collected data depends on the precise goals of the DE, on the hypotheses put to the test in it and on the conjectures made in the *a priori* analysis. However, particular attention is paid to the collection of data allowing the researcher to understand students' interaction with the milieu, and up to what point this interaction supports their autonomous move from initial strategies to the strategies aimed at, and to analyze devolution and institutionalization processes. Generally collected data include the students' productions including computer files when technology is used, field notes from observers, audios and, more and more, videos covering group work and collective phases. The data, collected during the realization are generally complemented by additional data (questionnaires, interviews with students and teacher, tests) allowing a better evaluation of the outcomes of the DE. During the realization, researchers are in the position of observers. It is important to point out that the realization often leads to make some adaptation of the design during the realization, especially when the DE is of substantial size, or from one realization to the next one when several realizations are planned in the research project. In that case, adaptations are of course documented together with the rationale for them and taken into account when the *a posteriori* analysis is carried out.

### 17.3.4 A Posteriori Analysis and Validation

*A posteriori* analysis is organized in terms of contrast with the *a priori* analysis. Up to what point do the data collected during the realization phase support the *a priori* analysis? What are the significant convergences and divergences and how can they be interpreted? What happened that was not anticipated and how can it be interpreted? Through this connection between *a priori* and *a posteriori* analyses, the hypotheses underlying the design are put to the test. It is important to be aware that there are always differences between the reference provided by the *a priori* analysis and the contingence analyzed in the *a posteriori* analysis. As observed above, the *a priori* analysis deals with generic and epistemic students, which is not the case for the contingence of the realization. Thus, the validation of the hypotheses underlying the design does not impose perfect match between the two analyses.

The analyses carried out are qualitative in nature and local, even when the researchers use statistical tools such as for instance implicative analysis for identifying dependences. In accordance with the theoretical foundation of DE, what the researcher looks at is the dynamic of a complex system, and he does so through the comparison of the observed dynamics with the reference provided by the *a priori* analysis, trying to make sense of similarities and differences. The precise tools used for that purpose depend on the research questions at stake and the data collected. There is no doubt however that these tools have evolved along the years, influenced by the global evolution of the field and also by the technological evolution. In general, researchers combine and triangulate different scales of analyses. They more and more include microscopic analyses taking into account the multimodality of the semiotic resources used by students and teachers that technology makes accessible today. To this should be added that, as mentioned above, the validation of the research hypotheses generally combines the analysis of data collected during the classroom sessions themselves and of complementary data.

### *17.3.5 The Nature of the Results*

It must be stressed that the results obtained through this methodology are mainly local, contextualized, and generally in form of existence theorems in their positive forms. For instance, in the research I developed about the teaching of differential equations in the mid-1980s (Artigue 1992, 1993), I used DE methodology to investigate the possibility of combining qualitative, algebraic and numerical approaches to the solving of ordinary differential equations in a university mathematics course for first year students. The research showed the possibility of organizing such a course in the French context, at that time, with the support of technological tools; it made clear what could be expected from such a course in terms of learning outcomes in this particular context and why. Beyond that, one important result was that a condition for the viability of the course was the acceptance by the didactical system of proofs based on specific graphical arguments, which violated the usual didactical contract[4] regarding proofs in Analysis at university. The difficulty of ensuring this acceptance out of experimental contexts and research control at that time hindered a large-scale dissemination of the developed didactical strategy, despite the fact that its robustness had been attested by realizations carried out with different categories of students. These results were certainly interesting but could not be generalized without precaution to another educational context. However, it would be unreasonable to consider that the results of this engineering work were limited to what we have summarized above.

As evidenced by the further use of this work by different researchers, the preliminary analyses carried out had a more general value, as well as the understanding gained on:

- the students' cognitive development in this area;
- the role played in it by the interaction between the quantitative and the qualitative, between algebraic and graphical representations;
- the affordances of technological tools for approaching the qualitative study of differential equations;

[4]The notion of didactical contract is a fundamental notion in the theory of didactical situations (Brousseau 1997). It expresses the mutual expectations, partly explicit but mainly implicit, of students and teacher regarding the mathematical knowledge at stake in a given situation. The rules of the didactic contract often become visible when they are transgressed by one actor or another.

- the characteristics of usual didactical contract regarding graphical representations and their didactical effects, especially the fact that proofs based on graphical arguments were not accepted.

Looking back at decades of DE research, what is evident indeed is that the results of DE research are far from being limited to the production and validation of didactical designs. DE research has also been a highly productive tool for understanding the functioning of didactical systems, and for identifying didactical phenomena. For decades, DE research has been an essential tool for the development of theoretical constructs paying justice to the complexity of the systems involved in the teaching and learning of mathematics.

What I have described here are the characteristics of the main form of DE: a research methodology based on the conception, experimentation and evaluation of a succession of classroom sessions having a precise mathematical aim. As already mentioned, this methodology has been extended to other contexts such as teacher education, to more open activities such as project work or modeling, and even to mathematical activities carried out in informal settings such as summer camps which obey a different form of contract, which Pelay (2011) defines as the *didactical and ludic contract.*[5] These extensions influence the content of preliminary analyses, but also what the design aims to control in terms of learning trajectories. The reference provided by the *a priori* analysis cannot exactly have the same nature, and this impacts the ways *a priori* and *a posteriori* analyses are contrasted.

### *17.3.6 Didactical Engineering and Design-Based Research*

I will finish this section by situating didactical engineering with respect to design-based research, using the definition of it provided in the Encyclopedia of mathematics education (Swan 2014, p. 148):

> Design-based research is a formative approach to research, in which a product or process (or 'tool') is envisaged, designed, developed and refined through cycles of enactment, observation, analysis and redesign, with systematic feedback from end users. In education, such tools might, for example, include innovative teaching methods, materials, professional development programs, and/or assessment tasks. Educational theory is used to inform the design and refinement of the tools, and is itself refined during the research process. Its goals are to create innovative tools for others to use, describe and explain how these tools function, account for the range of implementations that occur, and develop principles and theories that may guide future designs. Ultimately, the goal is *transformative*; we seek to create new teaching and learning possibilities and study their impact on teachers, children and other endusers.

[5] The didactical and ludic contract is defined as the set of rules that, implicitly or explicitly, fixes the respective expectations and regulate the behaviour of one educator and one or several participants, in a project combining ludic and learning aims.

This definition makes clear that design-based research and DE have some common methodological characteristics. Both methodologies are organized around the design of some educational tool; this design is informed by educational theory, but also contributes to its development. Moreover, both methodologies reject standardized validation processes based on the comparison of experimental and control groups through a pre-test/ post-test system. However, differences are visible. The global vision underlying design-based research is that of mathematics education as a design science whose aim is the controlled production of educational tools (Wittmann 1998; Collins 1992); the global vision underlying DE is of didactics of mathematics as a fundamental science, whose aim is the understanding of didactical systems and didactical phenomena, and which has also of course an applied dimension. This fundamental difference reflects in methodological characteristics. Design-based research is interventionist and iterative in nature, and the cyclic nature of its process is essential. Along the successive cycles, the design is refined but also experimented in wider contexts for studying how it functions with different categories of users, not involved in the design process, and what adaptations may be necessary for its large-scale use. Didactical engineering as a research methodology does not obey the same pattern. It is more a "phénoménotechnique" with the meaning given to this term by Bachelard (1937), a tool for answering didactical questions, for identifying, analyzing and producing didactical phenomena through the controlled organization of teaching experiments. This is the reason why the preliminary analyses with their different dimensions and the *a priori* analysis are a central part of the research work, and are given so much importance in the articles referring to this methodology. Of course, this does not mean that a DE used in research is built from scratch, but previous constructions when they exist are used to inform the *a priori* analysis; the process is not theorized as a cyclic process. Moreover, what concerns robustness and up-scaling is considered as a matter of development. I come back to this point in the last section of this chapter, but first illustrate the ideas developed up to now with two examples.

## 17.4 Two Particular Examples

### *17.4.1 A Paradigmatic Example: The Extension of the Field of Numbers by G. and N. Brousseau*

The first example I will consider is the paradigmatic example of the didactical engineering developed by N. and G. Brousseau, more than three decades ago, for extending the field of whole numbers towards rational and decimals (Brousseau and Brousseau 1987, English version: Brousseau et al. 2014). This engineering which ranges over 65 classroom sessions is a very big object when compared with usual constructions whose size is much more limited. I cannot enter into its very details but would like to show how this construction is characteristic of a DE piloted by the theory of didactical situations.

#### 17.4.1.1 Preliminary Analyses

This construction evidences first the importance attached to the preliminary analyses, and especially to their epistemological and didactical dimensions, the initial realizations having taken place in the COREM[6] where the institutional pressure was reduced. These analyses led Brousseau to question the usual educational strategy for extending the field of whole numbers. Usually indeed, the first step was the introduction of decimal numbers in connection with changes in units in the metric system, and fractions played a more marginal role. Emphasis was put on the continuity between the two systems of numbers (whole numbers and decimals), especially regarding the techniques for arithmetic operations, and the resistant cognitive difficulties that these strategies generated or reinforced were more and more evidenced by research. Brousseau made the hypothesis that, in their last years at elementary school, students were able to learn much more about rational and decimal numbers, for instance to differentiate the dense order of rational and decimal numbers from the discrete order of whole numbers, to appreciate the computational interest of decimal numbers and the possibility that this system offers for approaching rational numbers with arbitrary levels of precision. The didactical engineering developed aimed at testing the validity of this hypothesis with ordinary students.

#### 17.4.1.2 Conception and Analysis a Priori

The epistemological analysis carried out inspired the first macro-choice, in clear rupture with established practices: to extend first the field of numbers towards rational numbers, and then to particularize decimal numbers among these for the facilities they offer in terms of comparison, estimation and calculation. Regarding the introduction of rational numbers, another macro-choice was made linked to the identification of two different conceptions for rational numbers: a conception in terms of partition of the unit (1/n is then associated with the partition of one unit into n equal parts and the rational m/n represents m such pieces of the unit) and a conception in terms of commensurability, which corresponds to the search for a common multiple to two different magnitudes for instance two lengths (the ratio of two magnitudes is expressed by the rational m/n if m times the second one equals n times the first one). Generally didactical strategies privilege the first conception in the context of pizza parts or other equivalent contexts. This constitutes an easy entrance in the world of fractions but Brousseau hypothesized that it could contribute to the observed cognitive difficulties. This led him to explore the potential offered

[6] COREM was the Center for observation and research in mathematics education created by Brousseau in Bordeaux in 1973. An experimental elementary school was attached to this center, with very advanced means for systematic data collection and storage. The data collected there during more than 20 years are still studied by researchers, for instance, in the frame of the national project VISA (http://visa.ens.lyon.fr). Detailed information is accessible at the following url: http://guy-brousseau.com/le-corem/acces-aux-documents-issus-des-observations-du-corem-1973-1999/

by an entry in terms of commensurability, and to search for a fundamental situation attached to this conception: a situation that would oblige to consider multiples of magnitudes to compare them.

The problem posed to the grade 4 students was the following: how to compare the thickness of different sheets of paper? There is no doubt that this problem answers the condition just mentioned. The thickness of a sheet of paper cannot directly be measured with usual instruments but taking a sufficient number of such sheets one obtains something measurable. This problem being fixed, different choices must be done for defining a situation. Evident didactic variables are the number of types of paper to compare and their respective thickness. Anticipating that a basic strategy for students is to use their senses (sight and touch) for ordering the different types, it is important to have papers of close thickness invalidating perceptive strategies. Other choices concern, as mentioned above, the organization of the material milieu and the students' interaction with this milieu, the social organization of the classroom. In the organization adopted in this DE, the material milieu was made of piles of sheets of different thickness which often were very close and students worked in groups. First, they had to find a way of comparing the thickness of the sheets provided to their group, then in a second phase, after selecting one type of paper, to write a message allowing another group of students having the same types of paper to find the paper they had selected. These messages became then themselves an object of study: did the messages produced by the different groups solve the particular problem each group had to address, and, beyond that, did they provide a technique for solving the problem of comparison in a general way? We can see here a construction which takes into account the distinction made in the theory of didactical situations between three different functionalities of mathematical knowledge: for acting, for formulating, for proving. Their development obeys different dialectics and thus supposes different types of situations: *situations of action* in the first phase, *situations of formulation* in the second phase (in which the key for success is the quality of the specific language developed) and *situations of validation* in the third phase (in which what is at stake is the validity of assertions).

In an implicit way, the winning strategy in this situation uses the fact that the thickness of a pile is proportional to the number of sheets, which constitutes a reasonable model under certain limits, of course. In fact, the different couples of whole numbers attached to the same paper obtained through manipulations are not exactly proportional, which shows the distance that separates the real world from mathematical models. In observed realizations, this strategy systematically emerged through a-didactic interaction with the milieu. This emergence is certainly fostered by the presence of piles of paper in the material milieu. In the *a priori* analysis, it was expected that each type of paper would be eventually characterized by one or several couples of whole numbers that are nearly proportional, in reference to the manipulations carried out by the students. For instance, it could be 1 mm for 27 sheets in one case, 2 mm for 40 sheets in another case. Once such couples are obtained, as they do not necessarily correspond to the same number of millimeters or to the same number of sheets, if students are not allowed more manipulations, the success of the comparison relies on proportional reasoning. For a good functioning of the interaction with the milieu, it is thus necessary that some knowledge about proportional reasoning be part of the mathematical knowledge shared by students. In the *a priori* analysis, this knowledge is supposed from the generic student. For instance, if the task is to compare the types of paper corresponding to the two couples mentioned above, one can develop the following reasoning: for the first paper, 2 mm should correspond to 54 sheets, and 54 is more than 40, thus the second paper is thicker. For close thicknesses, comparison may be more delicate for the reasons mentioned above, and several exchanges of messages might be needed.

What is mathematically at stake in the solving of this problem is the ordered structure of rational numbers seen as couples of whole numbers or more appropriately families of such couples, and the conception attached is clearly the commensurability conception. As shown by the many realizations carried out, substantial work can be developed in this context about equality and order of rational numbers, students can progressively discover a good number of properties in a-didactic interaction with the successive milieus organized for them, validate them pragmatically using piles of paper, and then use piles of paper more metaphorically for supporting computations and reasoning. However, the mathematical knowledge built still remains attached to this specific context. There is no reason that the notations introduced by students and progressively refined for reasons of economy and efficiency are the conventional notations. This is the responsibility of the teacher to decide when to connect these classroom notations to the usual ones expected by the institution, and also to organize the decontextualization of knowledge through appropriate situations. Of course, in the DE, these steps are also carefully designed.

In this DE, the same context is then used for extending addition to these new numbers. However it does not allow to extend multiplication to rational numbers in a similar way. For this extension, the choice is made of privileging a conception of multiplication as an external operation in terms of linear application for which the well-known situation of the puzzle is the associated fundamental situation. With this new situation, it is also expected to make students face the epistemological obstacle of the additive model.

#### 17.4.1.3 Realization, Data Collection, *a Posteriori* Analysis, Validation and Further Outcomes

I cannot enter into more details in this DE structured in four main phases and invite the interested reader to consult the references mentioned above or the retrospective analysis provided by Brousseau and Brousseau (2007). In the description above, I have focused on the essential phases of design and *a priori* analysis of the methodology, trying to show how they were informed by the preliminary analyses and guided by the theory of didactical situations. The experimentations took place in the experimental school attached to the COREM, the sessions were observed by researchers according to specific guidelines and systematically video-recorded. The comparison of the *a priori* and *a posteriori* analyses, the complementary tests taken by the students, validated the hypotheses underlying the DE.

This DE was used year after year in the experimental school attached to the COREM. More than 750 students were exposed to it and its robustness was confirmed. However, as often stressed by Brousseau himself, it was never considered that it could be easily implemented in ordinary schools and become a standard teaching strategy. Moreover, the comparison of the successive dynamics attracted Brousseau's attention to the fact that the reproduction of the same situations, year after year, by a teacher generated what he called a phenomenon of obsolescence affecting the internal reproducibility of the DE. This phenomenon more globally raised the issue of the reproducibility of didactical situations that was theorized in further work (Artigue 1986).

It must also be stressed that this DE was in fact used for approaching a diversity of research questions, and for instance for investigating dependences between conceptions (Ratsimba-Rajohn 1982). In his doctoral thesis, indeed, Ratsimba-Rajohn, starting from the two strategies for associating a rational measure to a magnitude mentioned above (commensurability and partition of the unit), precisely differentiated these in terms of situations of effectiveness and mathematical knowledge engaged. This analysis led to the identification of a set of nine variables conditioning the effectiveness and cost of each strategy, depending on the type of task (game in the terminology used by the author, in line with the use of game theory in the theory of didactical situations). The author used this tool for investigating how students introduced to rational measures through the commensurability strategy, as was the case in the DE, could enrich their strategies by incorporating the partitioning strategy, *a priori* more intuitive and socially used. For that purpose, a sequence of three situations was designed as part of the DE. In the first situation, the commensurability strategy was extended to other magnitudes (length, weight, capacity); in the second situation, the tasks proposed were out of the domain of effectiveness of the commensuration strategy but could be solved using the partition strategy.[7] The goal of the third situation was to initiate the validation of equivalence of the two models when both strategies are effective. The corresponding lessons were implemented in two consecutive years. Students' strategies and their evolution were carefully documented. Different dynamics were identified. The most striking result was the difficulty that these students had at moving from commensuration strategies to partition strategies, even when commensuration was ineffective. These difficulties were confirmed by the evolution of students' answers at a test taken by the students before and after the teaching sequence in the first year of experimentation. All students significantly progressed in their answers to questions that favored the commensuration strategy or were neutral, only one student progressed on questions blocking the commensuration strategy. Difficulties met in using commensuration and efforts made for overcoming these difficulties in fact tended to reinforce this strategy and the associated conception of rational numbers; more was needed for

[7] This is the case for instance when pupils are asked to find a rational measure for a stick, a unit stick being provided, but the limitation of the physical space and material provided does not allow them to implement the strategy of commensuration.

integrating an alternative conception in terms of partition, despite the fact that it seemed *a priori* much more accessible than the commensuration conception.

### *17.4.2 An Example of Didactical Engineering Combining the Theory of Didactical Situations with Semiotic Perspectives*

The second example I consider is substantially different. It corresponds to a didactical engineering developed by Maschietto in her doctoral thesis (Maschietto 2002) on the transition between Algebra and Analysis. The goal of this DE was to explore the possibility of introducing students very early to the local/global game on functional objects fundamental in Calculus and Analysis, through the introduction of the derivative in terms of local linear approximation. The main hypothesis was that, through an appropriate use of the potential offered by symbolic and graphical calculators, this local/global game could be initiated already in high school, and that the idea of derivative could be built by the students as mathematization of a perceptive phenomenon. Another aim of this DE whose theoretical framework combined the theory of didactical situations and the theory of semiotic mediations (Bartolini Bussi and Mariotti 2008) was to analyze how gestures and metaphors (Arzarello and Edwards 2005; Lakoff and Nuñez 2000) contributed to the mathematization process and the cognitive development of students, as summarized by Maschietto (2008, p. 208):

> The research hypothesis is that the transformations of the graphical representation of a function through the use of zoom-controls and the experience of the perceptive phenomena of "micro-straightness" that these transformations provoke, can give rise to the formulation of some specific language, the construction of metaphors and the production of gestures and specific signs by the students. Our hypothesis is also that adequately exploited by thke teacher, these germs can lead to an entrance in the local/global game, fundamental in Calculus and Analysis hardly observed at high-school level.
>
> We find in this DE interesting variations from the standard case; they illustrate how, while maintaining the foundational values of this methodology, researchers can adapt it to their theoretical culture and needs. In this presentation, I will try to make clear how the theoretical combination at stake affects the methodological work.

#### 17.4.2.1 Preliminary Analyses

In this DE, we observe still the same attention paid to preliminary analyses. Maschietto developed a detailed analysis of the different perspectives that can be attached to a function: punctual, local, global, of the idea of local straightness, and of thinking modes in Analysis. Her epistemological analyses also aimed at understanding how, before the official introduction of the concept of limit, the language of infinitesimals could support the transition from Algebra towards Calculus, fostering the identification of rules for computations taking into consideration the respective order of magnitudes of the quantities involved.[8] From an institutional perspective, the DE was strongly constrained. Realizations could only be organized at the end of the school year in grade 11 in the Italian context, and in usual practices very few sessions were devoted to the topic. Moreover the use of calculators was usually limited in ordinary classrooms and that of symbolic calculators nearly non-existent. What was proposed was thus far apart from usual practices and would have been impossible to observe in naturalistic conditions. In fact, Maschietto worked with a teacher used to collaborate with researchers, but the institutional constraints limited the realization to a few sessions. Six sessions of 90 min were initially planned, but the thesis only analysed the three first sessions implemented in each of the three experimentations carried out.

Didactical analysis classically reviewed research carried out in that area which is substantial from the seminal work by Tall (1989). What this review showed nevertheless was that, even when the property of local straightness was put to the fore and the visualization potential of technology used for making students aware of it, the responsibility of the mathematization process was hardly devolved to them. Moreover, with few exceptions (see, for instance, Defouad 2000), the distance between what was seen on the screen of calculators or computers, or the equations provided by the calculator for tangent lines and the ideal mathematical objects was not necessarily questioned; thus the mathematization process was not fully developed. Research has also shown that when students enter Calculus, the idea of tangent is not new to them; they have coherent conceptions, geometric and algebraic ones, coming from the experience gained when working with circles. These conceptions lead to characterize the tangent to a curve as a line having a unique intersection point with the curve and staying on the same side of it, but not in terms of proximity (Castela 1995). This conception has to be questioned and as research also shows, usual teaching does not pay much attention to the reconstruction needed. Maschietto pointed out that, in Italy, these conceptions could be reinforced through the teaching of conics in grade 10. Her preliminary analyses also reviewed research developed on gestures and embodiment, as well as the metaphorical vision of mathematics developed by Lakoff and Nuñez (2000).

#### 17.4.2.2 Conception and Analysis a Priori

The conception phase of the DE relied on these preliminary analyses. In the first situation, students were asked to consider six different functions and after entering them in the calculator and getting their graphical representation in the standard window, to make successive zooms around particular points and to explore what happened.

They were also asked to sketch the initial representation and those obtained after two zooms and at the end of the exploration (when they had the feeling that the graphical representation was more or less stable), before moving to another function.

[8] For instance, taking into account the fact that, in the neighbourhood of 0, the order of magnitude of $x^2 + x$ is the order of magnitude of x.

The number and characteristics of the proposed functions and the selected points are evidently micro-didactical variables for this task. In the DE, the value of these was chosen so that students first met differentiable functions, then faced a function not differentiable at a point but having left and right derivatives (the function defined by $f(x) = -x^3 - 2|x| + 4$), a linear function and a function with a more complex behavior (the function defined by $f(x) = 4 + x.\sin(1/x)$ for $x \neq 0$ and $f(0) = 4$). It was hypothesized that the first examples would lead students to perceptively identify the local straightness phenomenon and to expect its emergence for further examples. The examples of non-differentiable functions would then oblige them to realize that there exist exceptions to this apparently common behavior and that these exceptions might present different characteristics. It was also expected that the dynamic process of zooming would make emerge discourses and metaphors able to support the further mathematization of the perceptive phenomena of local straightness. The drawings asked of the students were expected to be a useful support for this emergence, and for the substantial collective discussion at the end of the session. These drawings were also data to be used for the *a posteriori* analysis. Moreover, for each function two different points were selected for insisting on the local nature of the observed phenomenon. Students worked in pairs with one calculator for each pair and one common graphical production to deliver. This is a classical organization in DE for fostering verbal exchanges and making these accessible to researchers.

The aim of the second situation was the mathematization of this perceptive phenomenon. A differentiable function was selected, different from those already envisaged, and a particular point of its graphical representation. Students were asked to check its local behavior around this point and to find the equation of the line they had got on the screen. It was hypothesized that the different groups would manage the zooming process in different ways and stop it at different times, obtaining thus close but different lines. Using the Trace command or numerical values from the Table application of the calculator for getting coordinates of a second point of their line, they would thus get different equations. At this stage, it was planned that the teacher would collect and write on the blackboard all these equations and would launch a collective discussion. It was hypothesized that the view of the equations, close but different would lead students to consider all these lines as approximations of one ideal object: the tangent to the curve, whose equation they could conjecture from the equations written on the blackboard. The validation of this conjecture was not supposed to result from mere a-didactical interaction with the milieu. In the scenario for this session, it was planned that the teacher would ask students to find a common way of expressing the different computations and that, if this was not spontaneously proposed by them, she would introduce the idea of giving account of the commonalities between these different calculations through the use of a letter h representing the different small increments chosen by the students. From this point a collective computation was expected to lead to an equation for the line depending on h, but becoming the ideal equation when h was made equal to 0 (in some sense when infinite zooming was performed). This should allow the teacher both to institutionalize the definition of the tangent to a curve at a given point in terms of linear approximation, and the specific type of computation that allowed finding its equation. For this second situation, the characteristics of the function and of the point were the main micro-didactical variables of the task. In the DE, two different choices were successively made: a polynomial function of degree 2 and then one of degree 3, with simple coefficients and of a point whose coordinates were such that the ideal equation could be easily conjectured. Choosing a polynomial function and using the letter h in the symbolic computation resulted in the equation of the line described by a polynomial in h (after simplification by h), which made the reasoning easier. Choosing a polynomial of degree 3 made that the algebraic strategy known from these students for finding tangents to conics did no longer work. Once again students worked in pairs. In the third situation, it was planned to begin to consolidate the form of computation that had been introduced and also to connect this conception of the tangent in terms of approximation with those conceptions, geometric and algebraic, mentioned above, reinforced in grade 10, through the work with conics.

As mentioned above, it was hypothesized that during the three sessions, the students would combine gestures with the use of language and different semiotic representations for making sense of the situations and exchange with other students and the teacher. However, the exact forms these combinations would take, and the language that students were likely to introduce for qualifying local straightness was not anticipated. From that point of view, the DE had more an exploratory purpose.

Each session lasted 90 min and combined a phase of autonomous work by the students and a phase of collective discussion. Its *a priori* analysis was structured in the thesis around the following dimensions:

- the preparation of students' worksheets and analysis of them in terms of mathematical content, pre-requisites, didactical variables;
- the analysis of the role to be played by graphic and symbolic calculators in each phase of the session;
- the analysis of the work expected from the students, the anticipation of possible strategies and difficulties;
- the analysis of the work expected from the teacher in each phase of the session, and of the distribution of responsibility expected between students and teacher.

#### 17.4.2.3 Data Collection, *a Posteriori* Analysis and Validation

The collected data consisted of students' worksheets and productions, videos of one particular group and of collective phases, observation notes for different groups (two or three depending on the experimentation) according to guidelines defined in the analysis *a priori*. A test taken by students 2 weeks after the teaching experiment and a questionnaire filled by them regarding their participation in this experience were added. The semiotic perspective impacted the collection of data (those in charge of video-recording for instance tried to capture students' and teacher's gestures as much as possible) and the *a posteriori* analysis of the sessions.

The *a posteriori* analysis of each session combined two levels. The first level presented a global analysis of the session in its relation to the *a priori* analysis (regarding the scenario of the session, the distribution between group work and collective discussions, the strategies developed by the students and the main characteristics of their work, the difficulties observed, the teacher's role…). The second level was a fine-grained analysis of the data collected during the session elucidating the conceptualization processes at stake and their characteristics, through the role of the calculator, of metaphors, of discourse and gestures, of interactions between students during group work and between students and teacher.

We illustrate this methodological work by a few examples taken from the *a posteriori* analysis of the first session. For this session, the global analysis was structured around four dimensions: the scenario, the localization of the perspective, the emergence of the invariant and the role of the teacher. Regarding the localization of the perspective for instance, the main elements taken into account in this global approach were the characteristics of the graphical representations drawn by the different groups. A specific list of codes had been developed in the *a posteriori* analysis of the first experimentation, starting from students' productions. It was used again in the *a posteriori* analysis of the second and third experimentation. These codes showed the expected evolution of representations along the zoom process, but they also made evident the strength of the usual didactic contract regarding graphical representations of functions and the difficulty most students thus faced when the zooming process makes the axes disappear.

The analysis of data for the observed groups and for the collective discussion then combined different semiotic elements for clarifying the conceptualization processes at stake and the characteristics of the situation that fostered these conceptualizations (characteristics of the task, of the milieu and of social interactions). In particular, discourse, inscriptions and gestures were tightly connected in the analysis.

In the *a posteriori* analysis, the different levels of analysis for one particular session were then combined for testing the conjectures made in the *a priori* analysis regarding this particular session. The same type of *a posteriori* analysis was made for the three sessions, then the different results were synthesized and triangulated with those resulting from the analysis of the final test and questionnaire.

The following two quotations by Maschietto (2008) in which the author gives a synthetic vision of her research work, illustrate the form that these analyses have taken. The first quotation (pp. 215–216) regards the emergence of the linear invariant and an interesting phenomenon accompanying this emergence. This phenomenon was not anticipated in the *a priori* analysis but it had a positive effect on the dynamics of the situation.

Excerpt 1: DAL-DF-MA group (Exp_A)

| | |
|---|---|
| 15. DF | "Forward zoom" (*he carries out the 3rd ZoomIn*) |
| 16. DF | "Again" (*he carries out the 4th ZoomIn*) |
| 17. DF | "It becomes straighter and straighter" |
| 18. DF | "The drawing is the same as before. Even if the result is the same, we'll write it down".<br>After getting the representation in the standard window, DF does 2 ZoomIns |

DF "I want the other piece of function. It's still a line! Draw at least one axis" (*addressed to MA. DF carries out the 3rd ZoomIn*)
DF "We'll stop here because it stays the same".

In the pencil-and-paper environment (Fig. 17.1a), the linearity is emphasised by the use of a ruler to draw the graphical representation that appears on the calculator display on the third sheet (end of the exploration).
In other protocols (Exp_B and Exp_C), the students try to explain the end-point of their exploration, for example: "REASON WHY WE STOPPED CARRYING OUT THE ZOOMS → *The more we used the ZoomIn, the more the curve sector considered tended to become a line*". We observe here a dynamic language, that draws on the infinite approximation process.

In the protocols, there are two distinct phenomena, linked to the local point of view. The first regards the strength of the "straight" nature at a perceptive level. The second regards the interference of the global point of view with the local one. As far as the first phenomenon is concerned, the comments (for example, Excerpt 2) on the exploration of the corner (function y3[9]) highlight that at this stage the students have, in general, clearly identified the graphic phenomenon "it becomes straight using the zoom".

Excerpt 2: DAL-DF-MA group (Exp_A)
In all these cases the functions, even with the second zoom, are similar to a line with a gradient ≥0 but:

– y4[10] is similar to a line only after the 4th zoom [Note: at x = 1/pi]
– y3 is similar to two lines (one with m > 0 and the other with m < 0)

However, this recognition does not allow them to distinguish the situation of the function that is differentiable at the given point and that of the function having two different half derivatives and leading to a corner. In fact, these situations, mathematically different, are unified by their common "straightness" recognized at a perceptive level (Excerpt 2). The second function does not therefore represent a counter-example, unlike what is hypothesized in the a-priori analysis. Their distinction will only occur during the mathematization

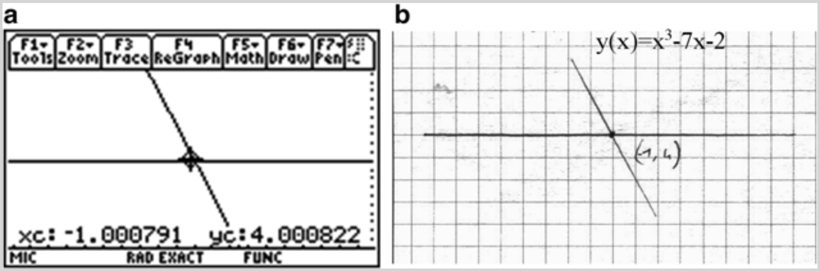


**Fig. 17.1** Window at the end of the exploration process (Exp_A)

[9] $y3(x) = -x^3 - 2|x| + 4$ at $x = 0$.
[10] $y4(x) = 4 + \sin(1/x)$ at $x \neq 0, = 4$ at $x = 0$

process of the linear invariant. The real counter-example is provided by the y4 function, the graphical representation of which, after subsequent zooms, is perceptively different. In this case there is no move from the “curve” category to the “straight” category, as happens for all the other functions.

The second quotation (pp. 217–218) shows the importance attached to gestures in the *a posteriori* analysis:

In accordance with the a-priori analysis, the activity presented to the students shows its potential for the production of gestures and metaphors. These appeared both during the communication inside the groups and during the collective discussions. The analysis of the students’ protocols and the discussions show that the conceptualisation of the zoom- controls, that supports the localisation of the view, appears through gestures that accompany the explanation of the exploration strategies and linguistic expressions that can be analysed in terms of metaphors.

A particularly representative example is the analysis of the gestures of one student, PM (Exp_A), while he is explaining the exploration of a graphical representation. The ZoomIn control is used in order to see some of the characteristics of the curve in a detailed way and is associated with a downward movement meaning an “entrance into the curve,” that corresponds with moving into the curve (ZoomIn gesture, Fig. 17.2a). The ZoomOut control, which is used to obtain a bigger curve and to study its characteristics better, is associated with an upward movement meaning an “exit from the curve” (ZoomOut gesture, Fig. 17.2b), which also corresponds with moving away from the curve. PM’s gestures lead the details of the curve to be interpreted as downwards and the overall curve as upwards. PM also creates a space in front of him for controlling these processes (the standard window of the calculator becomes a little rectangle that is constructed by his fingers, Fig. 17.2c).

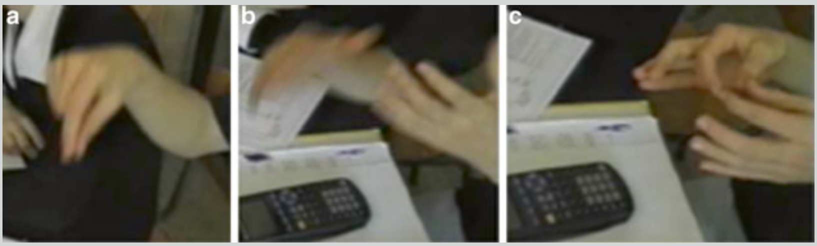


**Fig. 17.2** PA’s gestures (Exp_A): ZoomIn, ZoomOut, standard window

The reference to the ZoomOut control identifies the space under his eyes, while the palm of one hand is associated with the flat part that is obtained from the ZoomIn. In this way, PM has created his own space, which is suggested by the activity with the calculator, where the two different transformations of the curve can co-exist and be controlled.

The realization took place in three different classes as mentioned above, with some minor adjustments and evident regularities were observed. Globally the hypotheses mentioned above were confirmed despite the fact that it was not possible to cover all that had been planned and that, due to their previous experience with conics, some groups conjectured very early that the line was the tangent and privileged an algebraic strategy for finding its equation, persisting in that strategy with the polynomial of degree 3 in the second and third experiments. Some interesting and non-anticipated phenomena also occurred but they did not necessarily invalidate the *a priori* analysis. For instance, as shown in the first quotation above, it appeared that most students considered that straight lines and curves were objects belonging to different categories. This conception in fact helped them to consider that the linear representations they obtained by zooming were not exactly linear but just very close to a linear object, and that linearity could only be reached through an infinite succession of zooms. This helped them to make sense of the notion of tangent as an ideal object and of the computations carried out for finding its equation. This conception nevertheless also led them to think that the function admitting only left and right derivatives at a given point was not very different from the regular ones. This question was considered again later on once the derivative was properly defined. As expected also, gestures accompanied students' verbalizations and work, and the language and metaphors used by students showed evident embodiment. They introduced their own expressions for qualifying the phenomenon of local straightness, saying for instance that the functions were "zoomata lineare" at a particular point and these were accepted and used by the teacher. Validation of the DE did not just use the comparison of the *a priori* and *a posteriori* analysis of the sessions, but also the data from the questionnaire and interviews taken by the students after the completion of the process as mentioned above.

I cannot enter into more details here. The interested reader can find these in the references mentioned above. But I would like to stress a few points. According to the author, this methodological construction is a DE and I fully agree with this position, recognizing in it the fundamental features of DE presented above. This is nevertheless a construction sensibly different from that described in the first example. For instance, it is difficult to model the first situation as a game that students enter with basic strategies that they must make evolve towards winning strategies. Students are asked to stop their exploration when they have got the feeling that the graphical representations will no longer substantially evolve, which is a rather fuzzy condition. Moreover, if the situations are designed in order to ensure productive adidactical interaction with the milieu, in the construction of the situations an important role is given to collective discussions piloted by the teacher and to her mediations. These collective discussions are not just institutionalization phases. As evidenced by the *a posteriori* analysis, they play an essential role in the progression of knowledge beyond what has been achieved by each pair of students in the phase of autonomous work. In some sense, they play the role given in the theory of didactical situations to situations of formulation and of validation but they do not obey a similar organization; they are not supported by the same theoretical constructs. We can see here the effect of a combination of the theory of didactical situations and the theory of semiotic mediation. It shows us that, as a research methodology, DE can productively combine several theoretical approaches. Another close example is provided by the thesis by Falcade (2006) also combining the theory of didactical situations and the theory of semiotic mediation in an approach to functions using Cabri-Géomètre (see also Falcade et al. 2007).

## 17.5 Some Recent Developments of Didactical Engineering

### *17.5.1 Didactical Engineering and the Anthropological Theory of Didactics*

After considering these two examples, in the last part of the paper, we enter into some recent developments of didactical engineering, referring more precisely to the work carried out at the 2009 summer school.

As mentioned earlier, the anthropological theory of didactics has developed in the last decade a design perspective based on the idea of Programme of Study and Research (PSR in the following). At the 2009 summer school, Chevallard proposed to refund didactical engineering around this idea (Chevallard 2011). I will not follow him up to this point but would like to situate Chevallard's perspective with respect to the vision of DE that has been presented in the first sections of this chapter, and briefly explore some possible complementarities between these.

Through PSR, Chevallard wants to build a new epistemology opposing what he calls the "monumentalistic" doctrine pervading contemporary school epistemology (Chevallard 2006, in press). As explained by Chevallard (2006):

> For every praxeology[11] or praxeological ingredient chosen to be taught, the new epistemology should in the first place make clear that this ingredient is in no way given, or a pure echo of something out there, but a purposeful human construct. And it should consequently bring to the fore what its raisons d'être are, that is, what its reasons are to be here, in front of us, waiting to be studied, mastered, and rightly utilised for the purpose it was created to serve. (p. 26)

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[11] The notion of praxeology is central in the anthropological theory of didactics that considers that knowledge emerges from human practices and is shaped by the institutions where these practices develop. Praxeologies, which model human practices, at the most elemental level (punctual praxeologies), are defined as 4-uplets made of a type of task, a technique for solving this type of task, a discourse explaining and justifying the technique (technology), and a theory legitimating the technology itself.

In coherence with this vision, a PSR starts from the will to bring an answer to some generating question. In fact, at the 2009 summer school, Chevallard distinguished between different forms of PSR, and especially between finalized and open PSR. In finalized PSR, the main praxeologies aimed at are known. They correspond for instance to praxeologies aimed at by a given curriculum. The designer must found a question or a succession of questions which are able to generate the encounter of the corresponding types of tasks and the development of techniques and technological discourse constituting these praxeologies. This is done by a combination of study of existing works and inquiry processes. In open PSR, the situation is quite different. There is a generating question but the praxeological equipment needed for answering it is not *a priori* known; neither it is necessarily limited to mathematical praxeologies. This is for instance often the case in project work, and modeling activities.

Even in the case of finalized PSR, the proposed vision however is at some distance from the forms of DE mentioned above, especially in what concerns the milieu and its evolution. This is notably due to the place given to cultural answers to the question at stake in PSR. In the didactical schema that Chevallard proposes (Chevallard in press), a role is given to cultural answers or pieces of information accessible to the learners in the media and especially on the Internet. It is supposed that such cultural answers or pieces of information can enter the milieu on the initiative of teacher or students and that, duly studied and criticized, they should contribute to the elaboration of the expected answer to the question at stake. In the anthropological theory of didactics, this is encapsulated in the idea of *media-milieu dialectics*.

Differences with the classical vision of DE also concern more globally what the researcher ambitions to optimize and control in the design phase and consequently they affect the *a priori* analysis. This is especially the case for open PSR. For that case Chevallard denies the possibility of an *a priori* analysis. He thus introduces the idea of *analysis in vivo*, fully integrated into the inquiry work. This position can be questioned all the more as the publications of researchers working within this perspective show that they develop some form of *a priori* analysis to select questions having a strong generating power under the institutional conditions and constraints at stake. What is clear, however, is that, for such open PSR, in the *a priori* analysis researchers are more interested in investigating the didactical potential of the selected question, trying to make clear how its study can develop and generate new and interesting questions, motivate the study and progressive structuring of important praxeologies, than in the optimization of students' learning trajectories. In fact, the *a priori* analysis becomes an on-going process that develops and adjusts along the implementation phase of the DE. The doctoral thesis by Barquero (2009), (see also Barquero et al. 2008) analyzing the design and implementation of a PSR devoted to the modeling of population dynamics with undergraduate students provides a good example of such functioning.

There is no doubt that, from a DE perspective, the notion of open PSR makes it possible to address research issues attached to the functioning and viability of didactical forms more open than those usually addressed by existing DE such as project work and modeling activities. These didactical forms still have a marginal position in educational systems but they are also more and more encouraged as evidenced for instance by the number of European projects currently funded around inquiry-based education in mathematics and science.[12] As a research methodology, DE certainly needs some accommodation in order to cope efficiently with the research issues that emerge from this evolution, and also for taking into account the dramatic changes in access to information of the digital era. From this point of view, the design perspective offered by the anthropological theory of didactics seems promising.

### *17.5.2 Research and Development: Didactical Engineering of Second Generation*

The second evolution I would like to mention is that introduced by Perrin-Glorian (2011) who distinguishes between DE of first and second generation. In this chapter, we have considered DE from a research perspective focusing on its characteristics as a research methodology. We cannot forget nevertheless that from its emergence DE had the ambition to contribute both to research and development. In the historical review we mentioned the difficulties met at converting DE developed for research aims into useful educational resources. This problem is still not solved but the increase in our knowledge of teachers' representations and practices, and of possible dynamics for their evolution makes us better understand the difficulty of the enterprise. The distinction introduced by Perrin-Glorian directly addresses this issue and we consider it because it can also affect the vision of DE as a research methodology. Contrasting RDE and DDE (research didactical engineering and development didactical engineering), she compares the levels of theoretical controls in which these two forms of DE engage. She thus points out that even if in both cases the analysis of the mathematical knowledge at stake and of the students' knowledge, the definition of the situations and associated milieus are under theoretical control, for DDE much more flexibility is needed for preparing the adaptation to a diversity of contexts. The loss of control is even greater with regard to the role of the teacher while institutional constraints cannot be partly removed as is often the case in RDE. These considerations lead her to postulate that before trying to implement a DE product coming from research in ordinary classes, it is necessary to plan at least two different levels of DE, each one having specific aims: This is the whole process that she names DE of second generation.

At the first level, the goal is the theoretical validation of the situations of the DE (i.e. their capacity in producing the knowledge aimed at) and the identification of the fundamental choices of the DE, separating what is essential from what is linked to the particular context and could be changed, and adapted. The associated realization takes place in a rather protected environment and under the control of researchers as is the case for RDE.

[12] See the portal www.scientix.eu for information about these projects.

At the second level, the goal is the study of the adaptability of such validated situations to ordinary classrooms and teachers through the negotiation of the DE with teachers who have not been involved in the first phase. These negotiations and the transformations introduced by the teachers involved in this second phase are taken as objects of study together with their impact on the DE itself and its outcomes. It is expected that the results allow researchers to determine what concessions can be made in such negotiations, what should be preserved and why, and to identify what forms of control can be maintained.

As Perrin-Glorian points out, envisaging this second level modifies in fact the first level because it obliges researchers to move from a top-down conception of transmission of research results to an idea of adaptation much more dialectical. As she adds:

> The problem is no longer to control and disseminate engineering products coming from research but to determine the key variables, in terms of knowledge involved, piloting the didactical engineering that one wants to make a resource for ordinary teaching, and to study the conditions of their dissemination. (p. 69, our translation)

She then illustrates this vision by an example regarding the teaching of axial symmetry at the transition between elementary school and junior high school.

This reflection in fact points out that the transition from research to development needs specific forms of research, extending our view of the ways didactical engineering and educational research can be connected.

## 17.6 Conclusion

In this chapter, I have tried to present didactical engineering, focusing on its dimension of research methodology. To help readers make sense of this methodology, I have reviewed its history from its emergence in the early 1980s until now. I have tried to clarify its main characteristics and to show that this methodology, even if it has been shaped by the values and constructs of the theory of didactical situations, is a methodology that can be productively used beyond the frontiers of this theory, and is enriched by the different uses made of it. I have also tried to show that, as for many other constructs in educational research, didactical engineering is a living and dynamic concept which adapts to the evolution of the field, to the advances of educational knowledge, and to the evolution of the social and cultural contexts of mathematics education. I also hope to have made clear that this methodology, although flexible, imposes a systemic view of the field, a view of the classroom as a social organization, of learning as a combination of adaptation and acculturation processes and a particular sensitivity to the discipline and its epistemology.

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