Άσκηση 1 - 2ο Φυλλάδιο

$C: \vec{\sigma}(t) = (\cos t, \sin t, t), \; t \in [0, 2\pi]$

$\Rightarrow \vec{\sigma}'(t) = (-\sin t, \cos t, 1)$

$\rightarrow \| \vec{\sigma}'(t) \| = \sqrt{2}$

Άρα $\displaystyle \int_C f = \int_0^{2\pi} \cos t \cdot \sin t \, \sqrt{2} \, dt =$

$\displaystyle = \sqrt{2} \int_0^{2\pi} \cos t \, d(\cos t) = \frac{\sqrt{2}}{2} \left[\cos^2 t \right]_0^{2\pi} = 0$

$\square$

Άσκηση 2 – 2ο φυλλάδιο

C : τρίγωνο με κορυφές $A(1,0,0)$, $B(0,2,0)$, $\Gamma(0,0,3)$

$$\Rightarrow \begin{cases} \text{η πλευρά } \widehat{AB} \text{ βρίσκεται στο επίπεδο } z=0 \\ \text{-"-} \quad \widehat{B\Gamma} \quad \text{-"-} \quad \text{-"-} \quad \text{-"-} \quad x=0 \\ \text{-"-} \quad \widehat{\Gamma A} \quad \text{-"-} \quad \text{-"-} \quad \text{-"-} \quad y=0 \end{cases}$$

$$\Rightarrow \begin{cases} \int_{\widehat{AB}} f = \int_{\widehat{AB}} x\cdot y\cdot 0 = 0. \\ \int_{\widehat{B\Gamma}} f = \int_{\widehat{B\Gamma}} 0\cdot y\cdot z = 0 \\ \int_{\widehat{\Gamma A}} f = \int_{\widehat{\Gamma A}} x\cdot 0\cdot z = 0. \end{cases}$$

$$\Rightarrow \int_C f = 0.$$

□

Άσκηση 3 – 2ο φυλλάδιο

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = R^2\}$$

Παραμετροποίηση της S:

$$\vec{\Phi}(\theta,\varphi) = (R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi), \quad \begin{cases} \theta \in [0,2\pi) \\ \varphi \in [0,\pi] \end{cases}$$

(έτοιμο) $\Rightarrow \|\vec{\Phi}_\theta \times \vec{\Phi}_\varphi\| = R^2 \sin\varphi$

Επομένως: $$\int_S f = \int_0^{2\pi}\int_0^{\pi} (R\sin\varphi\cos\theta + R\sin\varphi\sin\theta + R\cos\varphi) R^2\sin\varphi \, d\varphi \, d\theta$$

$$= R^3 \int_0^{\pi}\int_0^{2\pi} (\sin^2\varphi\cos\theta + \sin^2\varphi\sin\theta + \cos\varphi\sin\varphi) \, d\theta \, d\varphi$$

$$= 2\pi R^3 \int_0^{\pi} \cos\varphi \sin\varphi \, d\varphi = 2\pi R^3 \left[\frac{\cos^2\varphi}{2}\right]_{\varphi=0}^{\varphi=\pi}$$

$$= 0$$

$\square$

Παρατήρηση: το $(0,0,0)$ είναι το κέντρο μάζας της S

Άσκηση 4 – 2ο φυλλάδιο

$$S=\{(x,y,z)\in\mathbb{R}^3 : z=4+x+y \text{ με } x^2+y^2\le 4\}$$

Παραμετρικοποίηση της S:

$$\vec{\Phi}(\rho,\theta)=(\rho\cos\theta,\ \rho\sin\theta,\ 4+\rho\cos\theta+\rho\sin\theta),\quad \begin{cases}\rho\in[0,2]\\ \theta\in[0,2\pi)\end{cases}$$

Είναι:

$$\left.\begin{aligned}\vec{\Phi}_\rho&=(\cos\theta,\ \sin\theta,\ \cos\theta+\sin\theta)\\ \vec{\Phi}_\theta&=(-\rho\sin\theta,\ \rho\cos\theta,\ -\rho\sin\theta+\rho\cos\theta)\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \vec{\Phi}_\rho\times\vec{\Phi}_\theta=\big(\cancel{+\rho\cos\theta\sin\theta}-\rho\sin^2\theta-\rho\cos^2\theta-\cancel{\rho\cos\theta\sin\theta},$$
$$\cancel{-\rho\sin\theta\cos\theta}-\rho\sin^2\theta-\rho\cos^2\theta+\cancel{\rho\sin\theta\cos\theta},\ \rho\big)$$

$$=(-\rho,-\rho,\rho)=\rho(-1,-1,1)\Rightarrow$$

$$\|\vec{\Phi}_\rho\times\vec{\Phi}_\theta\|=\rho\sqrt{3}.$$

Επομένως

$$\int_S f=\int_0^{2\pi}\int_0^{2}\rho(4+\rho\cos\theta+\rho\sin\theta)\rho\sqrt{3}\,d\rho\,d\theta$$

$$=\sqrt{3}\int_0^2\int_0^{2\pi}(4\rho^2+\rho^2\cos\theta+\rho^2\sin\theta)\,d\theta\,d\rho$$

$$=8\sqrt{3}\cdot\pi\left[\frac{\rho^3}{3}\right]_0^2=64\frac{\sqrt{3}}{3}\pi$$

$\square$

Άσκηση 5 – 2º Φυλλάδιο

$$S = \left\{ (x,y,z) \in \mathbb{R}^3 : z = \frac{x^2}{2} + \frac{y^2}{4} \text{ με } \frac{x^2}{9} + \frac{y^2}{36} \leq 1 \right\}$$

Παραμετρικοποίηση της S:

$$\vec{\Phi}(\rho, \theta) = \left(3\rho\cos\theta, 6\rho\sin\theta, \frac{9}{2}\rho^2\cos^2\theta + \frac{36}{4}\rho^2\sin^2\theta \right), \begin{cases} \rho \in [0,1] \\ \theta \in [0, 2\pi) \end{cases}$$

$$= 3\left(\rho\cos\theta, 2\rho\sin\theta, \frac{3}{2}(1+\sin^2\theta)\rho^2 \right)$$

Είναι:

$$\left.\begin{aligned} \vec{\Phi}_\rho &= 3\left(\cos\theta, 2\sin\theta, 3(1+\sin^2\theta)\rho\right) \\ \vec{\Phi}_\theta &= 3\left(-\rho\sin\theta, 2\rho\cos\theta, 3\rho^2\sin\theta\cos\theta\right) \end{aligned}\right\} \Rightarrow$$

$$\vec{\Phi}_\rho \times \vec{\Phi}_\theta = 9\Big(6\rho^2\sin^2\theta\cos\theta - 6\rho^2\cos\theta(1+\sin^2\theta), \\ -3\rho^2\sin\theta(1+\sin^2\theta) - 3\rho^2\sin\theta\cos^2\theta, 2\rho \Big)$$

$$= 9\left(-6\rho^2\cos\theta, -6\rho^2\sin\theta, 2\rho\right) \Rightarrow$$

$$\|\vec{\Phi}_\rho \times \vec{\Phi}_\theta\| = 18\cdot\rho\cdot\sqrt{1+9\rho^2}$$

Επομένως:
$$\int_S 1 = \int_0^{2\pi}\int_0^1 18\rho\sqrt{1+9\rho^2}\, d\rho\, d\theta$$
$$= 2\pi \cdot \int_0^1 \sqrt{1+9\rho^2}\, d(1+9\rho^2)$$
$$= \frac{4\pi}{3}\left(10^{3/2} - 1\right)$$

$\square$

Άσκηση 6 – 2ο Φυλλάδιο

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = \sqrt{x^2+y^2} \text{ με } 1 \leq x^2+y^2 \leq 2\}$$

Παραμετροποίηση της S:

$$\vec{\Phi}(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, \rho), \quad \rho \in [1,\sqrt{2}], \theta \in [0,2\pi)$$

Είναι: $\vec{\Phi}_\rho = (\cos\theta, \sin\theta, 1)$
$\vec{\Phi}_\theta = (-\rho\sin\theta, \rho\cos\theta, 0)$ $\Rightarrow$

$$\vec{\Phi}_\rho \times \vec{\Phi}_\theta = (-\rho\cos\theta, \rho\sin\theta, \rho) \Rightarrow$$

$$\|\vec{\Phi}_\rho \times \vec{\Phi}_\theta\| = \rho\sqrt{2}$$

Επομένως $\int_S f = \int_0^{2\pi}\int_1^{\sqrt{2}} \rho^3 \cdot \rho \cdot \sqrt{2}\, d\rho\, d\theta =$

$$= 2\sqrt{2}\cdot\pi \int_1^{\sqrt{2}} \rho^4 d\rho = 2\sqrt{2}\cdot\pi \frac{(\sqrt{2})^5 - 1}{5}$$

$$= 2\pi \cdot \frac{8-\sqrt{2}}{5} \quad \square$$

Άσκηση 7 – 2ο φυλλάδιο

$$S = \{(x,y,z) \in \mathbb{R}^3 : z = 2 - x^2 - y^2 \text{ με } x^2 + y^2 \leq 3\}$$

Παραμετρικοποίηση της S:

$$\vec{\Phi}(\rho,\vartheta) = (\rho\cos\vartheta, \rho\sin\vartheta, 2-\rho^2), \quad \rho \in [0,\sqrt{3}], \quad \vartheta \in [0,2\pi)$$

Είναι:

$$\left.\begin{aligned} \vec{\Phi}_\rho &= (\cos\vartheta, \sin\vartheta, -2\rho) \\ \vec{\Phi}_\vartheta &= (-\rho\sin\vartheta, \rho\cos\vartheta, 0) \end{aligned}\right\} \Rightarrow$$

$$\vec{\Phi}_\rho \times \vec{\Phi}_\vartheta = (+2\rho^2\cos\vartheta, -2\rho^2\sin\vartheta, \rho) \Rightarrow$$

$$\|\vec{\Phi}_\rho \times \vec{\Phi}_\vartheta\| = \sqrt{4\rho^4 + \rho^2} = \rho\sqrt{4\rho^2+1}$$

Επομένως:

$$\int_S f = \int_0^{2\pi}\int_0^{\sqrt{3}} \left(1 - (2-\rho^2)\right) \cdot \rho \cdot \sqrt{4\rho^2+1}\, d\rho\, d\vartheta$$

$$= 2\pi \int_0^{\sqrt{3}} (\rho^2 - 1) \cdot \rho \cdot \sqrt{1+4\rho^2}\, d\rho = \ldots$$ (υπολογίστε το μόνοι σας)