Mathematics for Teaching

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Reflecting a growing interest in mathematics education at all levels, many in the mathematics community have turned their attention to the mathematical preparation of prospective precollege teachers. Education researchers ([1], [2], for example) have documented striking differences in mathematical sophistication between teachers in the U.S. and teachers from other countries. A decade of thought and effort has produced several sets of specific recommendations for the improvement of the mathematical preparation of teachers [3], [4], [5]. And the NCTM document *Principles and Standards for School Mathematics* outlines some broad goals:

> Teachers need several different kinds of mathematical knowledge—knowledge about the whole domain; deep flexible knowledge about curriculum goals and about the important ideas that are central to their grade level; knowledge about how the ideas can be represented to teach them effectively; and knowledge about how students' understanding can be assessed. …This kind of knowledge is beyond what most teachers experience in standard preservice mathematics courses in the United States. ([6], p. 17)

Exactly what, if anything, needs to be changed in the mathematics courses teachers study in college? In this article I'll focus on the preparation of secondary (mainly high school) teachers, because most mathematics courses for future secondary teachers are taught in mathematics departments and because this is the grade span I know best. For the past three decades I've spent most of my time thinking about high school mathematics—teaching it, working with people who teach it or plan to teach it, and writing materials for it.

What Needs to Be Changed

Recommendations arise in an effort to improve things, so before I make mine I'd like to take a look, from the inside out, at what needs improving. In fact, most of what needs improving in high schools has nothing to do with the mathematical preparation teachers receive. Oppressive working conditions (five or more classes a day), the culture of schools, lack of resources, low pay and lack of respect in the community, top-heavy administrations, separation from the rest of the mathematics community, and government officials who propose political solutions to educational problems are at least as responsible as the current undergraduate curriculum is for the underperformance of mathematics programs. And it is, of course, a gross generalization to describe something as "a problem" when classrooms are as different as the people who teach in

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them. Still, there are some common features of almost every high school program I've seen that are very troublesome and that probably can be traced back to the kind of mathematics one learns as an undergraduate. Let me start by describing these; see if you recognize any of the following as habits undergraduates might pick up in your classes.

- *What it's all about.* In many schools, mathematics is perceived as an established body of knowledge that is passed on from one generation to the next. Instead of seeing the theorems of geometry, the formulas of trigonometry, and the methods of algebra as the *products* of doing mathematics, these artifacts are seen *as* the mathematics (see [7], [8], [9] for more on this). The questions people worried about and the struggles they went though trying to answer them almost never appear; instead, we see the results of the struggles, neatly packaged into pieces of boxed text. The mastery of any subject requires a thorough knowledge of basic facts, but when you see students studying for tests by learning the difference between the "$y = mx + b$ form" and the "two-point form", you know there's something wrong.
- *The "flatness" syndrome.* What really stands out in many of the classes I observe is that *everything* is on the same level. Vocabulary, notation, and convention are treated with as much importance as solving problems or proving theorems. So learning the definition of cosine is presented as the same kind of mathematics as proving the addition formulas. And the addition formulas are given the same emphasis as many other much more esoteric trigonometric identities. I've heard countless teacher-room discussions about whether or not a parallelogram is a trapezoid or whether the correct symbol for the whole numbers is $\mathbb{Z}^+$ or $\mathbb{N}$.
- *Worksheet-centered classes.* In many classes I've seen, the object of the game is to get through the work as quickly and as effortlessly as possible. The class routine consists of a continuous stream of delivering, completing, and collecting worksheets. The worksheets are usually of two types: forty identical calculations or a puzzle. In some cases the exercises or the puzzles are designed to make a point, but that point is hardly ever made in practice.
- *Watch-and-do pedagogy.* This style of teaching starts with the teacher working out a problem in detail. Next the students try it on an almost identical problem, mimicking what the teacher just did. This is followed by lots of practice on more nearly identical problems. The cycle then repeats until the end of class, where homework is assigned that consists of even more practice, just to cement things. And the problems often involve substituting numbers in formulas or applying some procedure, over and over.
- *Lemma-less curricula.* There's a deeply rooted notion in the precollege community that students need to have all the prerequisites for solving a problem before they can work on it. I was once part of a heated discussion about whether kids should be allowed to explore the theorem that the midpoints of a quadrilateral form the vertices of a parallelogram *before* they saw the theorem that a segment joining the midpoints of two sides of a triangle is half as long as the third side and parallel to it. I expected the need for that "midline" theorem would emerge in the course of the quadrilateral investigation, and it could be then posited as a lemma and treated later. This notion of assuming something for the time being to see where it gets you is completely foreign to most precollege curricula.
- *Learn it all in college.* Many teachers apply the above "know all you need before you start" philosophy to themselves. Instead of seeing their undergraduate studies as equipping them with the tools they need to become lifelong mathematics learners, they perceive that college is where you learn all the mathematics you'll ever need to teach the subject. The implications of this belief are quite profound. For example, most in-service programs for practicing teachers emphasize pedagogy, curriculum implementation, or other skills connected with the craft of teaching[1] much more than they focus on mathematical content—teachers learned "the math" when they were in college. Combined with the notion that mathematics is an established body of facts, this places many teachers in the frightening situation where their mathematical expertise is defined, not by what they can figure out, but by the facts they learned as undergraduates. With such a mindset, questions from students can be very intimidating.
- *The vertical disconnect.* Most teachers see very little connection between the mathematics they study as undergraduates and the mathematics they teach. This is especially true in algebra, where abstract algebra is seen as a completely different subject from school algebra. As a result, high school algebra has evolved into a subject that is almost indistinguishable from the precalculus study of functions. Another consequence is that, because individual topics are not recognized as things that fit into a larger landscape, the emphasis on a topic may end up being on some low-level application instead of on the mathematically important connections it makes. For example, look at what most curricula do

[1] *These skills are, of course, quite important. My point is that they receive vastly more attention than content in professional development programs.*

with DeMoivre's theorem. If it shows up at all, it is applied to finding small roots and powers of specific complex numbers. There are no connections to cyclotomy, constructibility of polygons, or algebraic and trigonometric identities.

These are generalities, and there are notable exceptions to every one of them. For example, most teachers of my vintage learned virtually nothing about technology in college but have, on their own, gained immense expertise in this area. And I know many teachers who are genuine mathematicians, who continue to study mathematics throughout their careers, and who have a real knack for helping students develop their own mathematical thinking. On the other hand, the problems just described are not the sole province of underprepared teachers or teachers working out of their areas of certification. I'm talking about teachers with undergraduate degrees in mathematics from good schools—teachers who may have studied with you. My only reason for making a list like this is that these are the kinds of serious problems that can, I think, be traced to the mathematics courses teachers take in college and that can be ameliorated by changing some things in the mathematical preparation of teachers.

I'm leaning on the "you teach the way you were taught" maxim knowing full well that it is an oversimplification. There are other influences on the mathematics that happens in high school classrooms. An important one is the materials (texts, tests, software, and so on) that teachers use. For a complex set of reasons having to do more with economics than education, large publishing houses have produced some truly abysmal curriculum materials that have, in turn, driven many of the problems on my list.

What Can Be Done?

Most of the problems described in the previous section are quite subtle. *Solutions will not come from rearranging the topics in a syllabus or by adding more topics to an already bloated undergraduate curriculum.* Making lists of topics that teachers should know when they graduate from college won't do it either. Yes, teachers need "the facts". But, in addition to learning something about its results, how can we help prospective teachers experience the *doing* of mathematics? How can we help them develop mathematical taste? How can we give them a sense for the really important questions that have led to breakthroughs? How can we help them develop the skill of seeing what to emphasize with their high school classes? How can we help teachers develop a passion for the discipline that lives on beyond college graduation? Here are some general principles that address these questions and that might be used to redesign part of the undergraduate experience for prospective teachers.

Make Connections to School Mathematics

One way to help practicing teachers develop the habit of "mining" the topics they teach for substantial mathematics is to make this habit explicit in undergraduate courses. For example, one could start with several seemingly different activities from high school texts, using them as springboards to the advanced mathematics that ties them together. I recently did just this with a group of middle and high school teachers as part of a field test for a professional development curriculum [10]. We started with activities they could use with their students: counting paths in "taxicab geometry" and counting ordered partitions of positive integers. Pascal's triangle came up in both investigations, so we looked for the underlying structural similarities between the two problems that would account for this. This led to a discussion of recursion, mathematical induction, the binomial theorem, computer algebra systems, partitions, and, finally, generating functions. By the end of the (ten-hour) course, every teacher had seen some new mathematics and had made new mathematical connections. I'm convinced that the distance we were able to travel depended crucially on the fact that the mathematics we discussed continually wove itself around the mathematics of middle and high school. And there are hundreds of units like this just waiting to be developed, sequences that start and end with school mathematics but that take students well into the realm of undergraduate mathematics along the way.

There are other ways of making connections to what teachers will teach. One promising idea is sometimes called a "shadow seminar". Prospective teachers in, say, a classical linear algebra class would attend a weekly seminar, designed perhaps by a mathematician, a practicing teacher, and a faculty member from a school of education. The seminar would shadow the course, showing how ideas in the course could be made tractable for precollege students or how they shed light on topics from precollege algebra and geometry. Variations on this idea abound. For example, Carole Greenes and colleagues from education and mathematics at Boston University are designing a "companion course" for prospective teachers that will help them tie together mathematics from abstract algebra, linear algebra, number theory, and the precollege curriculum.

Another way to connect with school mathematics is to show applications of mathematics to the craft of teaching. For example, at the University of Michigan Hyman Bass teaches a course on task design that looks at the design of student activities from both mathematical and psychological perspectives. Bass's course is an in-depth treatment of every aspect of designing student activities. But a more modest effort could be incorporated into mathematics courses for prospective teachers. For example, a great deal of classical undergraduate mathematics can be motivated by or applied to "meta problems" that teachers wonder about all the time as they invent problems for their students (see [11] for more on this theme). This mathematics could include questions like these:

- (How) can you generate Pythagorean triples?
- (How) can you generate integer-sided triangles with a 60° angle?
- If two polygons have the same area, (how) can you cut one up to get the other?
- (How) can you find three lattice points in the plane that determine a triangle with integer side-lengths?
- (How) can you generate cubic polynomials in $\mathbb{Z}[x]$ with distinct rational extrema and roots?
- (How) can you generate the graph of $y = \sin x$ with dynamic geometry software?

Applications of mathematics to science, finance, and sports have become staples of many undergraduate courses. Why not include applications of mathematics to the teaching of mathematics?

Look at Different Models of Classroom Organization

This is a tough one for me. I love going to talks, and a well-executed lecture is as enjoyable for me as a well-performed concert. But the fact is, most of my students don't see it that way. They need to work on problems *in class*. They need to bounce ideas off their classmates. Many of them have a very hard time learning something by listening to me present it. On the other hand, they learn a great deal by having me work on problems with them, and they love to make presentations of their own, especially if they have come up with something clever. This is not the way I was taught in college, and it took me several years as a teacher to figure out effective ways of getting my students to do mathematics. If we want prospective teachers to be effective *and* to teach the way they are taught, we should construct our undergraduate classrooms around what mathematicians *do* rather than around what they *say*.

I want teachers to see how mathematical results are obtained rather than how they are presented. We all know that these things are different. When we close the study door to begin work on a problem, what takes place is nothing like what's in the paper that appears a year and a half later. What we do behind closed doors is full of false starts, extensive calculations, experiments, and special cases. We reduce things to lemmas for which we have no proofs; suspend work on these lemmas and on other details until we see if they'll help at all; calculate and play with logical connections for hours; and hope that some order, some missing link, some new connection to old ideas will emerge out of all this immersion. We don't worry about truth or beauty or conviction or the purity of the deductive method. We just look for clues.

Some students can learn to do what we do behind the study door by studying what we say in public. *Many more can't.* That doesn't mean that they can't become teachers who have a real sense for mathematics. It just means that we have to be honest with them. We have to show them what mathematics is really about; we have to focus and organize our classes around the style of work used by mathematicians rather than around the results of that work. Students need to see and experience what goes into a solution as well as what comes out of it. They need to experience all the mucking around that happens *before* the polishing of the proof begins.

Put Mathematical Ideas into Larger Contexts

Undergraduate mathematics experiences for prospective teachers could do a great deal to make sure that topics from school mathematics are put in more general contexts so that prospective teachers get a chance to think about questions like "What is this an example of?" For instance, imagine undergraduate courses that explicitly discuss questions like these:

- Are there other functions that act like absolute value on the rational numbers? What does it mean to "act like absolute value"?
- Are there other reasonable ways to measure distance between points on the plane? What does that do to plane geometry?
- There's a Cauchy-Schwarz inequality in linear algebra and a Cauchy-Schwarz inequality in statistics. Is this just a coincidence?
- Geometric probability[2] suggests there's a connection between probability and area. Is there more than a superficial similarity?
- The books all say that Galois theory is about solving equations. What exactly do they mean by that?
- The formula for standard deviation looks a lot like the distance formula. Is that a coincidence?
- If you square a Gaussian integer, the real and imaginary parts are legs of a Pythagorean triple. Is *that* a coincidence?
- There's a characteristic equation in linear algebra and a characteristic equation used to solve difference equations. Are they connected?

I conjecture that if questions like these were taken up regularly, we'd see much more similarity between secondary and postsecondary subjects with the same name.

Build a Research Experience into Teacher Preparation

There are very few absolutes in education, but there's one thing of which I am absolutely certain: *The best high school teachers are those who have a research-like experience in mathematics.* I don't

[2] *Geometric probability in precollege mathematics centers around calculating the probability that a random point lands in a region by computing ratios of areas.*

mean research in the sense of producing new knowledge. Frontline problems take immense backgrounds. But the *methods* used by research mathematicians are widely accessible (my colleagues and I believe they are accessible to high school students). And working for an extended period of time on a hard problem that has no apparent approach or solution has profound effects on how one perceives the nature of the enterprise. Teachers who have done this type of research are much less likely to think of mathematics as an established body of facts than are teachers who have simply taken a set of courses. They are more likely to stay engaged in mathematics after they start teaching. They are used to looking for connections that don't live on the surface. And they are much more likely to organize their classes around large investigations rather than low-level exercises. There are many rich areas of investigation on which undergraduates can embark without a huge amount of machinery—"no threshold, no ceiling" problems that allow them to work as mathematicians for a piece of their undergraduate experience. An ideal teacher preparation program combines the kind of orchestrated assimilation of the main results in mathematics that you get in courses with the much messier unstructured explorations that come from working with a mentor and grappling with a research project.

Some Examples

Over the years many people have implemented subsets of these and other suggestions for improving teacher preparation. Here are just a few that I know about. I'm sure there are many others. Perhaps the *Notices* is one place where discussions of innovative approaches in this area can appear.

My first example is almost thirty years old. In 1972 I took a summer course from the late Ken Ireland (of Ireland-Rosen fame [12]) at Bowdoin College. The course had a tremendous influence on my life. Ireland's premise was that there are dozens of famous mathematical results that are part of the "folklore" of elementary mathematics. Some of these results go back to the Greeks, some come from arithmetic and number theory, and some involve classical algebra or analysis (examples include the impossibility of certain constructions with straightedge and compass and the transcendence of π). They are folklore in the sense that most teachers know the *statements* of the results, but few know about their proofs or even their history. The course was constructed to fix that; in six very intense weeks Ireland helped us develop the mathematical backgrounds to understand the proofs of these famous facts, how they are connected to each other, and where they sit in the history of mathematics. The course was based on about 200 problems, handed out in waves and supplemented by daily lectures that were anything but didactic and that changed forever my own thoughts about effective teaching. It's amazing how ahead of its time this course was. We were encouraged to work together on the problems (and even to pass in joint solutions), an idea unheard of at the time but that has become a centerpiece of the current reform movement in mathematics education. The lectures were designed to give us the connections we'd need to make progress on the problems, not to provide us with templates we could use to solve them. This emphasis on "teacher as coach" is another basic ingredient to today's reform movement, but I remember how disquieting it was to many of us in 1972. That summer introduced me to the notion that one could design a substantial mathematics course organized around topics that underlie school mathematics, using a teaching style that lets struggling with problems lead the way.

My next example is old too. It traces its pedigree back forty years to the Ross program at Ohio State. Glenn Stevens and David Fried, together with Marjory Baruch and Steve Rosenberg, have transported an enhanced version of the program to Boston University, where it has been running as PROMYS (Program in Mathematics for Young Scientists) for over a decade. For almost that long, teachers (prospective and practicing) have been attending the program, spending six weeks for each of two summers immersed in number theory and other mathematics. The program for teachers is a perfect example of how immersion in mathematics is effective teacher preparation and professional development. It centers around three activities:

1. *The courses.* Teachers take an intense six-week number theory course along with the high school students in the program. Like Ireland's course, it centers around problems: carefully orchestrated sets are passed out, graded, and passed back in cycles of one day. Stevens gives a morning lecture, but, as he says, "I see to it that I never discuss a topic unless people have struggled with it for three days."
2. *The research experience.* The number theory experience is carefully planned, and the problem sets lead to specific results. To complement this, each teacher works with three students on a research project. Participants are given suggestions ("circles of ideas", as they are known at PROMYS) from which they design and investigate a project for the entire summer. The projects allow easy entry but can (and often do) lead teachers and students into very new and advanced territory.[3]

[3] *For examples of the kinds of projects used in the program, see Michelle Manes's project for high school students and their teachers at* http://www2.edc.org/makingmath/.

3. *The academic-year seminars.* Between the two summers teachers attend five all-day seminars at Education Development Center in which they work on translating the PROMYS experience into classroom practice. We look for connections between the PROMYS topics and school mathematics. We also look for ways to implement the teaching style of allowing students to explore a topic before teachers present it.

Teachers find the PROMYS experience overwhelming. They see what it's like to do real mathematics—always being at the edge of what you understand, having much more to do than you can possibly finish, and seeing hints of mysterious connections emerge almost out of nowhere. Extensive support structures are provided to deal with the frustration. There are undergraduate and graduate students on hand to act as counselors and to lead problem sections, and the faculty is always around for help. By the end of three weeks most participants turn a corner, and they start working as real mathematicians. They develop a view of what mathematics is about that will help teachers avoid the problems I described earlier.

The preceding two examples have the luxury of not being confined to the academic-year schedule of semesters and courses. But even within these constraints, some very creative approaches are being tried out:

- Hung-Hsi Wu has developed a set of principles for designing courses for mathematics majors who do not intend to go to graduate school. These principles include many devices designed to address the problems I described earlier (for a complete list see [13]):

  – Make explicit connections between the topics in the course and topics in elementary mathematics.

  – Place topics in their historical contexts.

  – With surveys and exposition, place topics in their broader mathematical contexts.

  – Give motivation at every opportunity.

  Wu has implemented his complete set of principles with some success in several upper-division courses for mathematics majors at Berkeley. His algebra course, for example, goes a long way to help prospective teachers make connections between school algebra and algebra as a mathematical discipline.
- Bill McCallum describes (personal communication) a program at the University of Arizona, developed, with accompanying texts [14], [15], by David Gay and Fred Stevenson. Undergraduates who decide to teach high school replace abstract algebra and analysis with two innovative courses:

  – *Introduction to Number Theory and Modern Algebra* starts with the natural numbers (and elementary number theory) and builds to a construction of the real numbers. Along the way students look at the theory of decimal expansions, find rational points on algebraic curves, and study various ways to represent real numbers. The course then gives students an extensive selection of projects that deal with topics ranging from Fibonacci numbers to continued fractions.

  – *Topics in Geometry* reads to me like a catalogue of what every high school teacher needs in his or her back pocket. Measure and measurement, polyhedra, shortest path problems, kaleidoscopes, symmetry, and isoperimetric problems are all treated.
- Joe Rotman has developed a course (and an accompanying text [16]) that he uses at the University of Illinois to give students a broad sense of what doing mathematics is about. The course helps students develop self-confidence in writing proofs by starting with results about binomial coefficients that can be proved by mathematical induction. Rotman believes that what students need is not experience with truth tables but time to develop the skill of producing convincing proofs, and he spends a great deal of time in his course having students read, critique, develop, and present proofs. The core course also treats convergence of sequences and the algebra of complex numbers. Additional topics include Pythagorean triples, parametrizing the circle and conic sections, a discussion of π leading to a proof of its irrationality, and the cubic and quartic formulas.

These ideas, courses, and examples are all very different, but they share several features that are essential to preparing quality high school teachers:

- They have a coherent design and a focussed goal.
- They show mathematics as something you do rather than something you memorize.
- They emphasize (and are explicit about) the thinking and habits of mind employed by working mathematicians.
- They bring students into the culture of mathematics—a culture with its own history, aesthetics, elegance, and even humor.
- They focus on the interactions among the students and the instructor.
- Problems precede abstractions, experience precedes axiom systems, and student thinking is at the center of the work.

Every time I make a list like this, I wonder why this shouldn't be the kind of undergraduate mathematics experience *everyone* gets.

It's customary, when designing mathematics curricula in the U.S., to concentrate on lists of topics to be covered. We've become quite good at that, and very reasonable lists can be found elsewhere. But my contention here is that such lists are bound to be ineffective if we don't find ways to communicate the spirit of doing mathematics to the people who plan to teach it.

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