CLASSIFYING PROCESSES OF PROVING

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This paper outlines a preliminary classification of the kinds of justifications that students offer in mathematical contexts, i.e., their "proof schemes." The classification is based primarily on the work of students during teaching experiments and individual interviews, with secondary and post-secondary students. The dominant, natural proof schemes of most students--even university mathematics majors--are not ones accepted in the mathematical community as giving mathematical proofs. Transformational proof schemes are viewed as essential for advancing beyond these schemes; teaching experiments with university students suggest that many students can make pleasing progress toward expecting and giving acceptable mathematical proofs.

Many researchers have given attention to different aspects of the learning and teaching of proof (e.g., Bell, 1976; Chazan, 1993; Fischbein and Kedem, 1982; Hanna, 1990; Martin and Harel, 1989; Senk, 1985; Yerushalmy, 1993). These indicate that the ideas of proof are difficult for students to learn, at least as they are currently taught. A quote from Poincaré summarizes our position toward the teaching and learning of proof in mathematics:

> It is difficult for a teacher to teach something which does not satisfy him entirely, but the satisfaction of the teacher is not the unique goal of teaching; *one has at first to take care of what is the mind of the student* and what one wants it to become. [via Artigue, 1994; emphasis added]

Accordingly, we have been concerned with attempting to determine what *is* in the minds of students, when proof comes up in mathematics. Others have had the same concern. For example, Chazan (1993) noted that U.S. high school geometry students were skeptical that a deductive proof assured that there were no counterexamples to the assertion proved, and that a proof was only further evidence that a conjecture is true. Fischbein and Kedem (1982) found that among students in an Israeli program of studies involving the greatest concentration on mathematics only about one-third of the students who had endorsed a statement and its proof realized that further checks of specific instances would be superfluous.

Our approach has been to focus on justifications, and to view a mathematical proof as the type of justification that is usually accepted by the mathematical community. During interviews, mostly of university students in courses for mathematics majors, we have attempted to determine what sorts of justifications convince them, and what sorts of justifications they would offer in order to convince others. During teaching experiments with university students, the thrust has been to help students refine their own ideas about what constitutes justification in mathematics.

Categories of Proof Schemes

The notion of "proof scheme" has been useful to us. Proving (or justifying) a statement includes two aspects: ascertaining (convincing oneself) and persuading (convincing others). An individual's proof scheme consists of whatever constitutes ascertaining and persuading <u>for that person</u>. Hence, a proof scheme is idiosyncratic and naturally can vary from time to time and from context to context, even within mathematics. It is important to note that "proof" as used in "proof scheme" need not connote "mathematical proof." The teaching experiments have had the intent to identify and alter students' proof schemes, and the interviews to test the sufficiency of the classification. The categories as currently conceived fall into three major classes. In a few cases the labels for the proof schemes are tentative, so the reader should rely not so much on the labels as on the brief descriptions and illustrations.

<u>The External Conviction Proof Schemes</u>

The earmark of the external conviction proof schemes is that justifications hinge on such external features as the endorsement of an authority (the authoritarian proof scheme), the form of the argument (the ritual proof scheme), or meaningless manipulations of symbols (the symbolic proof scheme).

<u>The Authoritarian Proof Scheme</u>. When students are not concerned with the question of the burden of proof, and their main source of conviction is a statement given in a textbook, uttered by a teacher, or offered by a knowledgeable classmate, they are exhibiting the authoritarian proof scheme. When asked how they might convince someone of a particular result, statements like "I would try to find it in a book" or "I think my professor said it, so it should be in my notes" would be offered under this proof scheme. The value of proofs may even be questioned, perhaps because in so much of the mathematics that the student has experienced the emphasis has been on the results, with little or passing attention to the reasoning processes used to arrive at those results. In the teaching experiments, where "why" is a routine expectation as well as "how," students have gradually become less unquestioningly accepting of assertions deliberately made by the instructor to test their willingness to accept the mere word of the "authority."

All this is not to say that accepting the word of an authority is all bad, of course. Even noted mathematicians are no doubt on many occasions willing to accept a result without examining the details of a proof. Rather, it is the attitude of helplessness in the absence of an authority, or the view that justifications are valueless, that handicap the students with an authoritarian proof scheme.

<u>*The Ritual Proof Scheme*</u>. *Martin and Harel (1989) examined whether students' judgments of an argument are influenced by its appearance in the form of a* mathematical proof--the ritualistic aspects of proof--rather than the *correctness* of the argument. They presented students with a false argument to a given mathematical statement and then examined the students' evaluations of that argument. They found that "many students who correctly accepted a general-proof

verification did not reject a false-proof verification; they were influenced by the appearance of the argument--the ritualistic aspects of the proof--rather than the correctness of the argument" (p. 49).

The "ritual proof" misconception, however, does not have to manifest itself in such a severe behavior as the judging of mathematical arguments on the basis of their appearance only. For example, on many occasions during the beginning period of a teaching experiment, either in a class discussion or in a personal exchange, students have asked whether a certain justification is considered a proof. When asked to explain the motivation for their question, the students indicated that although they are convinced by the justification, they have doubts whether it counts as a mathematical proof, for "it does not look like a proof." Typically such doubts are raised when the justification is not communicated via mathematical notations and does not include symbolic expressions or computations, even though the argument itself is quite sound by the usual mathematical standards; it is just that the argument does not "look" like a proof.

The Symbolic Proof Scheme. Justifications which use symbols as if they possess a life of their own *without reference to their possible functional or quantitative relations to the situation* characterize the symbolic proof scheme. The power of symbols is well known, but when symbols are empty of meaning, or bear no relationship to the situation for which the symbols were introduced, their use can be counterproductive. For example, it is not uncommon for linear algebra students to interpret the inverse of matrix A as the fraction 1/A, and attempt to reason about the inverse matrix as though it were a fraction.

Perhaps the most devastating consequence of the symbolic scheme is the common behavior of approaching problems without first comprehending the problem situation and its task. It is not unusual to find that immediately after reading the problem, many students begin their solution with some sorts of symbol manipulation of any expressions involved, with little or no time spent on comprehending the problem statement. Students' actions take place quite haphazardly without a clear purpose and without the formation of a coherent image of the problem situation. So, for example, many attempt a solution without knowing the meaning of some of the terms used in the problem statement, and many others are unable to articulate the exact task they were to accomplish. For these university students, the symbol manipulation rules they acquired in their earlier school years define the essence of their mathematical world: quantitative comprehension and sense making, wherein lie the value in representations by symbols, were absent from this world.

The Empirical Proof Schemes

These proof schemes are based solely on examples. As with the authoritarian proof scheme, reasoning based on examples is not entirely bad. Mathematicians value examples highly (see, e.g., Halmos, 1985). Psychologists nowadays note that natural concept formation is based on examples, and sometimes on rather special examples (Medin, 1989). But as Sfard points out, mathematics students must become "sufficiently mature in the mathematical culture" to appreciate the role of definitions in mathematics (1992, p. 47). A similar maturity in the mathematical culture should lead to an awareness of the tentative nature of results suggested by examples.

Inductive Proof Scheme. When students ascertain themselves and persuade others about the truth of a conjecture by evaluating their conjecture in one or more specific cases, they are said to possess an inductive proof scheme. Every teacher has likely observed the dominance of this proof scheme among students, and research corroborates this observation. For example, Chazan (1993) has observed the existence of the inductive proof scheme among U.S. high school students. Martin and Harel (1989) found that more than 80% of their preservice elementary teachers considered inductive arguments to be mathematical proofs. Even with mathematics majors, who presumably are more sophisticated than the high school students or the preservice elementary teachers, the inductive proof scheme is common.

The Perceptual Proof Scheme. This proof scheme fits, for example, many geometric justifications that might be given by younger students. The perceptual proof scheme is based solely on visual or tactile perceptions. For example, a student may examine an isosceles triangle and decide that the base angles are congruent just by visual examination. Older students might be convinced that the medians of a triangle are concurrent by looking at several computer-generated examples, and they might attempt to convince others by showing them similar examples.

The Theoretical Proof Schemes

The Transformational Proof Schemes. The general characterization of these schemes is that students' justifications attend to the generality aspects of a conjecture and involve mental operations that are goal oriented and intended-anticipatory. They are the foundation for all theoretical proof schemes. Here is an example of transformational reasoning from a case study of a fourth-grader (by GH):

> I asked Ed to think of a triangle with two equal angles and describe what he thought the relationship between the sides opposite them. Ed responded almost instantly that the two sides must be equal. I asked Ed to explain to me how he had arrived at this conclusion. Using his hands to describe the triangle, Ed said something to the effect that if one angle (he puts one forearm horizontally and moves the second forearm diagonally to it) is equal to the other angle (switches between the forearms' positions), then the two sides (he puts the two forearms diagonally to form a triangle) are equal. When I continued to press Ed for more explanation, he went on to say: If you launch a rocket from this side (pointing to his right elbow and moving his right forearm diagonally to indicate the direction of the rocket) and at the same time you launch another rocket from this side (pointing to his left elbow and moving his left forearm diagonally to indicate the direction of the other

rocket), the two rockets will collide and explode at the vertex of the triangle. Their parts will go down exactly in the middle of the triangle and make two little triangles. When you put these triangles together, one on top of the other (he lines up his two hands along the two little fingers and then opened and closed them several times), these two sides would be equal.

Notice the generality of the thinking and its basis in mental operations. Note also that the thinking could easily be turned into the common mathematical proof (since Ed was a fourth-grader, he was not asked to do this).

The transformational proof schemes classification includes three types of transformational proof schemes. Ed's justification illustrates a spatial-images proof scheme, which in general is characterized as a transformational proof scheme in which the context of the justification is of images from spatial intuition.

"Symbolic-transformational proof scheme" is our current label for an encapsulated transformational proof scheme that has become a heuristic in devising mathematical justifications. Repeated applications of transformational proof schemes, if reflected upon, can potentially result in the formation of proof heuristics. Hence, a symbolic-transformational proof scheme is a proof heuristic abstracted from the experience of applying transformational proof schemes. Here is an example, in which an older student transforms the given algebraic expressions into mental images related to graphs:

> Prove that for $x \geq 0$, $\log(x+1) \leq x$. He first converted this inequality into its equivalent $x+1 \leq e^x$, then he said: "Both functions [$x+1$ and e^x] are increasing but e^x goes faster. At zero they are equal, so e^x must be greater."

This student then translated this thinking into a more standard mathematical proof form.

One particularly important example of the symbolic-transformational proof scheme is this: To prove or refute a certain conjecture, the conjecture is represented algebraically and symbol manipulations on the resulting expressions are performed, with the intent to derive relevant information that deepens one's understanding of the conjecture and potentially leads to its proof or refutation. In this activity, the individual does not necessarily form conceptual images for some or all of the algebraic expressions and relations that result in the process. It is only at critical stages in this process--viewed as such by the individual--that the person intends to form such images.

The third transformational scheme is the constructional proof scheme. In the constructional proof scheme a students' doubts are removed by actual construction of objects, as opposed to mere justifications of the existence of the objects. For example, in justifying that the inverse of a square matrix is unique (when it exists), some linear algebra students have preferred a justification in which the inverse of a matrix is constructed, step-by-step, to the usual assume-there-are-two-and-show-they're-equal proof, even though the proof by construction was based on a 2x2 case with numerical entries. The students, most of whom realized the drawbacks of arguments based on specific numerical cases, regarded the argument with the specific case as a generic argument and preferred it because "you can see how it works."

The Structural Proof Schemes. The general characterization of these schemes is that they are special transformational proof schemes in which conjectures and facts are representations of situations from different realities that share a common structure. The structure is characterized by a collection of accepted facts. There are three subcategories, which will be described only briefly here. It is important to keep in mind that these must be transformational in nature; otherwise there is the danger of resorting to rote memory in settings where they could be used. The postulational proof scheme is a structural proof scheme in which the structure is characterized by a collection of permanently accepted facts. This scheme is essential in studying the theory of vector spaces, for example. The spatial-postulational proof scheme is a postulational proof scheme whose realities are based in intuitions of space. The postulates in Hilbert's *Grundlagen der Geometrie*, for example, could provide the characterization with which to justify statements in geometry. Finally, the axiomatization proof scheme is a structural proof scheme in which the structure is characterized by a collection of tentatively accepted facts. This scheme is essential in studying questions of consistency, independence, completeness, etc.

Implications

The symbolic and ritual proof schemes, grounded as they are in meaningless symbol manipulation or surface features, have nothing to recommend them; perhaps with a greater emphasis on the giving of justifications instruction can help students to avoid them. Students must be educated to value and to want to know justifications; the source of the results, not just the results, must be emphasized. The authoritarian proof scheme, on the other, hand is a two-edged sword. In the culture of schools or of knowledge acquisition, it can be valuable. The concern is to move away from a complete reliance on it and its suffocating effect on the giving of justifications. For example, in the teaching experiments, a conjecture was no longer labelled "theorem," simply because the label "theorem" seemed to reduce the students' effort, willingness, and even the ability for some students to justify the conjecture. The label "theorem" apparently rendered the relationship into something to obey rather than to reason about. The use of small groups, in which there is no obvious authority figure, seems to foster more openness to evaluating justifications; there the student is a more genuine partner in justifying statements than in a teacher-led justification. The empirical proof schemes, with their roots in everyday thinking, are important and valuable. The inductive proof scheme is so strong, however, that instruction must deliberately combat it to show its defects.

To become "sufficiently mature in the mathematical culture" or to progress toward Poincaré's what-one-wants-the student's-mind-to-become, it is clear that a student must move beyond the external and empirical proof schemes. Of greatest