rocket), the two rockets will collide and explode at the vertex of the triangle. Their parts will go down exactly in the middle of the triangle and make two little triangles. When you put these triangles together, one on top of the other (he lines up his two hands along the two little fingers and then opened and closed them several times), these two sides would be equal.

Notice the generality of the thinking and its basis in mental operations. Note also that the thinking could easily be turned into the common mathematical proof (since Ed was a fourth-grader, he was not asked to do this).

The transformational proof schemes classification includes three types of transformational proof schemes. Ed's justification illustrates a spatial-images proof scheme, which in general is characterized as a transformational proof scheme in which the context of the justification is of images from spatial intuition.

"Symbolic-transformational proof scheme" is our current label for an encapsulated transformational proof scheme that has become a heuristic in devising mathematical justifications. Repeated applications of transformational proof schemes, if reflected upon, can potentially result in the formation of proof heuristics. Hence, a symbolic-transformational proof scheme is a proof heuristic abstracted from the experience of applying transformational proof schemes. Here is an example, in which an older student transforms the given algebraic expressions into mental images related to graphs:

> Prove that for $x \geq 0$, $\log(x+1) \leq x$. He first converted this inequality into its equivalent $x+1 \leq e^x$, then he said: "Both functions [$x+1$ and e^x] are increasing but e^x goes faster. At zero they are equal, so e^x must be greater."

This student then translated this thinking into a more standard mathematical proof form.

One particularly important example of the symbolic-transformational proof scheme is this: To prove or refute a certain conjecture, the conjecture is represented algebraically and symbol manipulations on the resulting expressions are performed, with the intent to derive relevant information that deepens one's understanding of the conjecture and potentially leads to its proof or refutation. In this activity, the individual does not necessarily form conceptual images for some or all of the algebraic expressions and relations that result in the process. It is only at critical stages in this process--viewed as such by the individual--that the person intends to form such images.

The third transformational scheme is the constructional proof scheme. In the constructional proof scheme a students' doubts are removed by actual construction of objects, as opposed to mere justifications of the existence of the objects. For example, in justifying that the inverse of a square matrix is unique (when it exists), some linear algebra students have preferred a justification in which the inverse of a matrix is constructed, step-by-step, to the usual assume-there-are-two-and-show-