

verification did not reject a false-proof verification; they were influenced by the appearance of the argument--the ritualistic aspects of the proof--rather than the correctness of the argument" (p. 49).

The "ritual proof" misconception, however, does not have to manifest itself in such a severe behavior as the judging of mathematical arguments on the basis of their appearance only. For example, on many occasions during the beginning period of a teaching experiment, either in a class discussion or in a personal exchange, students have asked whether a certain justification is considered a proof. When asked to explain the motivation for their question, the students indicated that although they are convinced by the justification, they have doubts whether it counts as a mathematical proof, for "it does not look like a proof." Typically such doubts are raised when the justification is not communicated via mathematical notations and does not include symbolic expressions or computations, even though the argument itself is quite sound by the usual mathematical standards; it is just that the argument does not "look" like a proof.

The Symbolic Proof Scheme. Justifications which use symbols as if they possess a life of their own *without reference to their possible functional or quantitative relations to the situation* characterize the symbolic proof scheme. The power of symbols is well known, but when symbols are empty of meaning, or bear no relationship to the situation for which the symbols were introduced, their use can be counterproductive. For example, it is not uncommon for linear algebra students to interpret the inverse of matrix A as the fraction $1/A$, and attempt to reason about the inverse matrix as though it were a fraction.

Perhaps the most devastating consequence of the symbolic scheme is the common behavior of approaching problems without first comprehending the problem situation and its task. It is not unusual to find that immediately after reading the problem, many students begin their solution with some sorts of symbol manipulation of any expressions involved, with little or no time spent on comprehending the problem statement. Students' actions take place quite haphazardly without a clear purpose and without the formation of a coherent image of the problem situation. So, for example, many attempt a solution without knowing the meaning of some of the terms used in the problem statement, and many others are unable to articulate the exact task they were to accomplish. For these university students, the symbol manipulation rules they acquired in their earlier school years define the essence of their mathematical world: quantitative comprehension and sense making, wherein lie the value in representations by symbols, were absent from this world.

The Empirical Proof Schemes

These proof schemes are based solely on examples. As with the authoritarian proof scheme, reasoning based on examples is not entirely bad. Mathematicians value examples highly (see, e.g., Halmos, 1985). Psychologists nowadays note that natural concept formation is based on examples, and sometimes on rather special