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SOCIOMATHEMATICAL NORMS, ARGUMENTATION, AND AUTONOMY IN MATHEMATICS

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This paper sets forth a way of interpreting mathematics classrooms that aims to account for how students develop mathematical beliefs and values and, consequently, how they become intellectually autonomous in mathematics. To do so, we advance the notion of *sociomathematical* norms, that is, normative aspects of mathematical discussions that are specific to students' mathematical activity. The explication of sociomathematical norms extends our previous work on general classroom social norms that sustain inquiry-based discussion and argumentation. Episodes from a second-grade classroom where mathematics instruction generally followed an inquiry tradition are used to clarify the processes by which sociomathematical norms are interactively constituted and to illustrate how these norms regulate mathematical argumentation and influence learning opportunities for both the students and the teacher. In doing so, we both clarify how students develop a mathematical disposition and account for students' development of increasing intellectual autonomy in mathematics. In the process, the teacher's role as a representative of the mathematical community is elaborated.

For the past several years, we have been engaged in a research and development project at the elementary school level that has both pragmatic and theoretical goals. On one hand, we wish to support teachers as they establish classroom environments that facilitate students' mathematical conceptual development. On the other hand, we wish to investigate children's mathematical learning in the classroom. The latter involves developing perspectives that are useful for interpreting and attempting to make sense of the complexity of classroom life. The purpose of this paper is to set forth a way of interpreting classroom life that aims to account for how students develop specific mathematical beliefs and values and, consequently, how they become intellectually autonomous in mathematics, that is, how they come to develop a mathematical disposition (National Council of Teachers of Mathematics, 1991). To that end, we focus on classroom norms that we call sociomathematical norms. These norms are distinct from general classroom social norms in that they are specific to the mathematical aspects of students' activity. As a means of introducing and elaborating the theoretical discussion in this paper, we present episodes from a classroom that we have studied extensively. The episodes have been

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selected for their clarifying and explanatory power and are not meant to be exemplary or reflect ideal classroom practice.

There is a reflexive relationship between developing theoretical perspectives and making sense of particular events and situations. The analysis of the particular constitutes occasions to reconsider what needs to be explained and to revise explanatory constructs. Conversely, the selection of particulars to consider reflects one's theoretical orientation. Thus, particular events empirically ground theoretical constructs, and theoretical constructs influence the interpretation of particular events (Erickson, 1986). This interdependence between theory and practice is reflected throughout this paper.

THEORETICAL PERSPECTIVE

Our theoretical perspective is derived from constructivism (von Glasersfeld, 1984), symbolic interactionism (Blumer, 1969), and ethnomethodology (Leiter, 1980; Mehan & Wood, 1975). We began the project intending to focus on learning primarily from a cognitive perspective, with constructivism as a guiding framework. However, as we attempted to make sense of our experiences in the classroom, it was apparent that we needed to broaden our interpretative stance by developing a sociological perspective on mathematical activity. For this purpose, we drew on constructs derived from symbolic interactionism (Bauersfeld, Krummheuer, & Voigt, 1988; Blumer, 1969; Voigt, 1985, 1989) and ethnomethodology (Krummheuer, 1983; Mehan & Wood, 1975). We were then able to account for and explicate the development of general classroom social norms. These same constructs proved critical to our development of the notion of sociomathematical norms. As will be seen throughout, constructs that proved particularly relevant are the interactive constitution of meaning, from symbolic interactionism, and reflexivity, from ethnomethodology. A detailed discussion of the coordination of psychological and sociological perspectives is beyond the scope of this paper and can be found in Cobb and Bauersfeld (1995).

Bauersfeld (1988) and Voigt (1992) have elaborated the relevance of interactionist perspectives for mathematics education research. A basic assumption of interactionism is that cultural and social processes are integral to mathematical activity (Voigt, 1995). This view, which is increasingly accepted by the mathematics education community (Cobb, 1990; Eisenhart, 1988; Greeno, 1991; Resnick, 1989; Richards, 1991), is stated succinctly by Bauersfeld (1993).

> [T]he understanding of learning and teaching mathematics ... support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using mathematics, or better: a culture of mathematizing as a practice. The many skills, which an observer can identify and will take as the main performance of the culture, form the procedural surface only. These are the bricks for the building, but the design for the house of mathematizing is processed on another level. As it is with cultures, the core of what is learned through participation is *when* to do what and *how* to do it. Knowledge (in a narrow sense) will be for nothing once the user cannot identify the adequateness of a situation for use. Knowledge, also, will not be of much help, if the learner is unable to flexibly relate and transform the necessary elements of knowing into his/her actual situation. This is to say, the core effects as emerging from the participation in the culture of a mathematics classroom will appear on the metalevel mainly and are "learned" indirectly. (p. 4)

In this view, the development of individuals' reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meanings.

Voigt (1992) argues that, of the various theoretical approaches to social interaction, the symbolic interactionist approach is particularly useful when studying children's learning in inquiry mathematics classrooms because it emphasizes the individual's sense-making processes as well as the social processes. Thus, rather than attempting to deduce an individual's learning from social and cultural processes or vice versa, it treats "subjective ideas as becoming compatible with culture and with intersubjective knowledge like mathematics" (Voigt, 1992, p. 11). Individuals are therefore seen to develop their personal understandings as they participate in negotiating classroom norms, including those that are specific to mathematics.

As we will demonstrate, the construct of reflexivity from ethnomethodology (Leiter, 1980; Mehan & Wood, 1975) is especially useful for clarifying how sociomathematical norms and goals and beliefs about mathematical activity and learning evolve together as a dynamic system. Methodologically, both general social norms and sociomathematical norms are inferred by identifying regularities in patterns of social interaction. With regard to sociomathematical norms, what becomes mathematically normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants. At the same time these goals and largely implicit understandings are themselves influenced by what is legitimized as acceptable mathematical activity. It is in this sense that we say sociomathematical norms and goals and beliefs about mathematical activity and learning are reflexively related.

SOCIAL AND SOCIOMATHEMATICAL NORMS

In the course of our work, we have collaborated with a group of second- and third-grade teachers to help them radically revise the way they teach mathematics. Instruction in project classrooms typically consists of teacher-led discussions of problems posed in a whole-class setting, collaborative small-group work between pairs of children, and follow-up whole-class discussions where children explain and justify the interpretations and solutions they develop during small-group work. The instructional tasks and the instructional strategies used in project classrooms have been developed during several yearlong classroom teaching experiments. In general, the approach we have taken reflects the view that mathematical learning is both a process of active individual construction (von Glasersfeld, 1984) and a process of acculturation into the mathematical practices of wider society (Bauersfeld, 1993).

Our prior research has included analyzing the process by which teachers initiate and guide the development of social norms that sustain classroom microcultures characterized by explanation, justification, and argumentation (Cobb, Yackel, & Wood, 1989; Yackel, Cobb, & Wood, 1991). Norms of this type are, however, general classroom social norms that apply to any subject matter area and are not unique to mathematics. For example, ideally students should challenge others' thinking and justify their own interpretations in science or literature classes as well as in mathematics. In this paper we extend our previous work on general classroom norms by focusing on normative aspects of mathematics discussions specific to students' mathematical activity. To clarify this distinction, we will speak of *sociomathematical* norms rather than social norms. For example, normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant in a classroom are sociomathematical norms. Similarly, what counts as an acceptable mathematical explanation and justification is a sociomathematical norm. To further clarify the subtle distinction between social norms and sociomathematical norms we offer the following examples. The understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm. Likewise, the understanding that when discussing a problem students should offer solutions different from those already contributed is a social norm, whereas the understanding of what constitutes mathematical difference is a sociomathematical norm.

In this paper we first document the processes by which the sociomathematical norms of *mathematical difference* and *mathematical sophistication* are established. Next, we illustrate how these sociomathematical norms regulate mathematical argumentation and influence the learning opportunities for both the students and the teacher. We then consider how the teacher and students interactively constitute what counts as an *acceptable mathematical explanation and justification*. In the process, we clarify how the teacher can serve as a representative of the mathematical community in classrooms where students develop their own personally meaningful ways of knowing.

Issues concerning what counts as different, sophisticated, efficient, and elegant solutions involve a taken-as-shared sense of *when* it is appropriate to contribute to a discussion. In contrast, the sociomathematical norm of what counts as an acceptable explanation and justification deals with the actual *process* by which students contribute. Because teachers with whom we collaborated were attempting to establish inquiry mathematics traditions in their classrooms, acceptable explanations and justifications had to involve described actions on mathematical objects rather than procedural instructions (Cobb, Wood, Yackel, & McNeal, 1992). For example, describing manipulation of numerals per se would not be acceptable. On the other hand, it was not sufficient for a student to merely describe personally real mathematical actions. Crucially, to be acceptable, other students had to be able to interpret the explanation in terms of actions on mathematical objects that were experientially real to them. Thus, the currently taken-as-shared basis for mathematical communication served as the backdrop against which students explained and justified their thinking. Conversely, it was by means of mathematical argumentation that this constraining background reality itself evolved. We will therefore argue that the process of argumentation and the taken-as-shared basis for communication were reflexively related.

Further, we will argue that the construct of sociomathematical norms is pragmatically significant, in that it clarifies how students in classrooms that follow an inquiry tradition develop mathematical beliefs and values that are consistent with the current reform movement and how they become intellectually autonomous in mathematics. Therefore, in keeping with the purpose of this paper, we limit our discussion to classrooms that follow an inquiry tradition. Nevertheless, sociomathematical norms, such as what counts as an acceptable mathematical explanation and justification, are established in all classrooms regardless of instructional tradition.

To clarify the theoretical constructs developed in this paper, we have selected examples from a second-grade classroom in which we conducted a yearlong teaching experiment. Data from the teaching experiment include video recordings for all mathematics lessons for the entire school year and of individual interviews conducted with each student in the class at the beginning, middle, and end of the school year. Field notes and copies of students' written work are additional data sources.

THE PROCESS OF DEVELOPING SOCIOMATHEMATICAL NORMS

As part of the process of guiding the development of a classroom atmosphere in which children are obliged to try to develop personally meaningful solutions that they can explain and justify, the teachers with whom we have worked regularly asked if anyone had solved a problem in a *different way*. It was while we were analyzing teachers' and students' interactions in these situations that the importance of sociomathematical norms, as opposed to general social norms, first became apparent. We will use the notion of *mathematical difference* to clarify and illustrate how sociomathematical norms are interactively constituted in the classroom.

In project classrooms, as in most mathematics classrooms, there were no pregiven criteria for what counted as a *different* solution. Instead, the meaning of what constituted *mathematical difference* was negotiated by each teacher and his or her students through their interaction. For their part, the teachers were themselves attempting to develop an inquiry form of practice. They did not have prior experience asking children to generate their own solution methods or explain their own thinking and, therefore, had little basis for anticipating methods the children would suggest. In the absence of predetermined criteria, the children had to offer solution methods without knowing in advance how they would be viewed by the teacher. Consequently, in responding to the teacher's requests for different solutions, the students were simultaneously learning what counts as mathematically different and helping to constitute what counts as mathematically different in their classroom. It is in this sense that we say the meaning of *mathematical difference* was interactively constituted by the teacher and the children. The teacher's responses and actions constrained the students' developing understanding of mathematical difference and the students' responses contributed to the teacher's developing understanding.

The following episode clarifies and illustrates how the teacher initiates the interactive constitution of mathematical difference.

Example 1: The number sentence 16 + 14 + 8 = _____ has been posed as a mental computation activity.

Lemont: I added the two 1s out of the 16 and [the 14] ... would be 20 ... plus 6 plus 4 would equal another 10, and that was 30 plus 8 left would be 38.

Teacher: All right. Did anyone add a little *different*? Yes?

Ella: I said 16 plus 14 would be 30 ... and add 8 more would be 38.

Teacher: Okay! Jose? *Different*?

Jose: I took two tens from the 14 and the 16 and that would be 20 ... and then I added the 6 and the 4 that would be 30 ... then I added the 8, that would be 38.

Teacher: Okay! It's almost similar to—(Addressing another student) Yes? *Different*? All right.

Here, the teacher's response to Jose suggests that he is working out for himself the meaning of *different*. However, because he does not elaborate for the students how Jose's solution is similar to those already given, the students are left to develop their own interpretations. The next two solutions offered by students are more inventive and are not questioned by the teacher.

Rodney: I took one off the 6 and put on the 14 and I had ... 15 [and] 15 [would be] 30, and I had 8 would be 38.

Teacher: Yeah! Thirty-eight. Yes. *Different*?

Tonya: I added the 8 and the 4, that was 12.... So I said 12 plus 10, that would equal 22 ... plus the other 10, that would be 30—and then I had 38.

Teacher: Okay! Dennis—*different*, Dennis?

By participating in exchanges such as this, the children learned that the teacher legitimized solutions that involved decomposing and recomposing numbers in differing ways but not those that were little more than restatements of previously given solutions. At the same time, the teacher furthered his pedagogical agenda by guiding the development of a taken-as-shared understanding of what was mathematically significant in such situations.

The next example further highlights the subtle and often implicit negotiation of the meaning. In this case, we see a student taking the initiative as he protests that a solution should not have been offered because, in his view, it was not different from one already given.

Example 2: The problem 78 – 53 = _____ was written on the chalkboard and posed as a mental computation activity.

Dennis: I said, um, 7 and take away 50, that equals 20.

Teacher: All right.

Dennis: And then, then I took, I took 3 from that 8 and then that left 5.

Teacher: Okay. And how much did you get?

Dennis: 25....

• • •

Teacher: Ella?

Ella: I said the 7, the 70, I said the 70 minus the 50 ... I said the 20 and 8 plus 3,... Oh, I added, I said 8 minus the 3, that'd be 5.

Teacher: All right. It'd be what?

Ella: And that's 75 ... I mean 25.

Dennis: (Protesting) Mr. K., that's the same thing I said.

Dennis's final comment serves two functions. With regard to the class discussion, it contributes to the negotiation of the meaning of *mathematical difference*. For the observer, it shows he understands that in this class it is not appropriate to offer an explanation that repeats a previously described decomposition and recombination of numbers. The notion of when it is appropriate to contribute to the discussion was taken as shared by at least some members of the class.

The preceding example clarifies that, in addition to regulating their participation in discussion, the sociomathematical norm of what constitutes mathematical difference supports higher-level cognitive activity. To respond as he did, Dennis had to compare his and Ella's solutions and judge the similarities and differences. In doing so, his solution became an object of his own reflection. In general, the teacher's requests for different solutions initiate a change in the setting from solving the problem to comparing solutions. In the latter setting the children's activity extends beyond listening to, and trying to make sense of, the explanations of others to attempting to identify similarities and differences among various solutions. Such reflective activity has the potential to contribute significantly to children's mathematical learning.

In the classroom studied, developing a taken-as-shared understanding of what counts as a *sophisticated solution* or an *efficient solution* was less explicit than an understanding of what counts as a *different solution*. For example, in this classroom the teacher rarely asked if anyone had a *more sophisticated* way or a *more efficient* way to solve a problem and never explicitly referred to one solution as better than another. Nevertheless, in any classroom, children are well aware of the asymmetry between the teacher's role and their role. The teacher necessarily represents the discipline of mathematics in the classroom (Voigt, 1995). Consequently, the teacher's reactions to a child's solution can be interpreted as an implicit indicator of how it is valued mathematically. For instance, in Example 1, many children may have interpreted the teacher's enthusiastic response ("Yeah!") following Rodney's solution as an indication that this solution was favored. However, because the issue did not become an explicit topic of conversation, the children were left to decide in what sense the solution was special. Events of this type are occasions for the children to infer what aspects of their mathematical activity the teacher values. In the process, the teacher both elaborates his own interpretative stance toward mathematics and inducts students into that stance.

The following episode, which occurred within the first few weeks of the school year, clarifies how mathematical discourse can advance as the teacher and students interactively constitute a taken-as-shared understanding of what is valued mathematically.

Example 3: The task is to figure out how many chips there are in a double-tens frame that has four red chips on the left frame and six green chips on the right frame (see Figure 1). The image was flashed on the overhead screen several times and then left off while the children figured out their solutions. The episode begins after several children have already given solutions that involve counting by ones.

Travonda: You could say, um, um, it's 6 on this side (pointing to the right frame) and take one from that side (pointing to the right frame) [and] put it on the red side and...

Teacher: *Listen to her!*

Travonda: And [you] would have 5 plus 5.

Teacher: *All right! Do you understand what she* [said]. *I like that!* She said (pointing to the screen) if we were to take one of these green and put it over here with, with the four [red chips] we'd have what?

Class: Five.

Teacher: Five. And this would leave five here (pointing to the right tens frame) and you could say 5 plus 5. *That's good.*

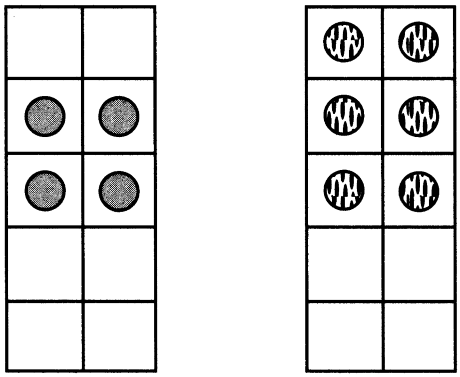

Figure 1. Double tens-frame task.

Even though the teacher did not indicate in what sense the solution Travonda gave was desirable, his expression of delight left no doubt that, in his view, this solution was special. As Voigt (1995) notes, such judgments serve an important function in supporting students' mathematical learning by making it possible for them to become aware of more conceptually advanced forms of mathematical activity while, at the same time, leaving it to them to decide whether to take up the intellectual challenge. Students can develop a sense of the teacher's expectations for their mathematical learning without feeling obliged to imitate solutions that might be beyond their current conceptual possibilities. In this case, several children took up the challenge of attempting to give solutions that they infer might also qualify as special. The episode continued as follows:

Chad: You, you can put the four [red chips] on that [right] side and you would make 10.

Teacher: *Yeah! I like that.*

Teacher: (To class) Chad says put these four (pointing to the red chips) over here (pointing to the blank spaces on the right frame) and that would make how many?

Class: Ten.

Teacher: Ten. *Okay, that's good.* Yeah?

Greg: Two plus 2 is four (pointing to the red chips) and 2 plus 2 is 4 (pointing to four green chips) and that's 8, and 2 more is 10.

Teacher: *Right. Do you understand what he said?* (The teacher repeats the solution for the class.)

John: You could do 7 plus 3 and then that would be 10.

Teacher: *I like that.*

Our observations indicated that all of the solutions that followed the teacher's enthusiastic response to Travonda's solution were novel for this class. For his part, the teacher continued to call attention to the solutions, indicating both that he wanted the other children to understand them and that he valued them. In the process, the sophistication both of individual children's thinking and of the mathematical discourse advanced. In Example 3, for instance, the solutions children offered became more sophisticated after the teacher indicated that he valued Travonda's solution. In this case, by sophisticated, we mean that the solutions went beyond counting by ones and involved constructing numerical relationships and developing alternative ways of combining elements of the two collections. John's comment, "*You could do* 7 plus 3 and then *that would be* 10," illustrates that children engaged in this type of extended activity. His language of "could do" and "that would be" suggests that, rather than reporting the way he initially solved the problem, he may be describing a relationship that he now realizes he could have used to solve the problem.

INFLUENCE OF SOCIOMATHEMATICAL NORMS ON MATHEMATICAL ARGUMENTATION AND LEARNING OPPORTUNITIES

We noted earlier that additional learning opportunities arise when children attempt to make sense of explanations given by others, to compare others' solutions to their own, and to make judgments about similarities and differences. Analysis of the children's activity shows that they constructed increasingly sophisticated concepts of ten, partitioned and recomposed two-digit numbers flexibly, and developed ways of talking about their mental activity using the standard language of tens and ones (Yackel, Cobb, & Wood, in press). Further, by explaining and justifying different solutions, the teacher and students established taken-as-shared meanings for tens and ones. In the process, these became experientially real mathematical objects (Davis & Hersh, 1981) for almost all of the children in the class.

The negotiation of sociomathematical norms gives rise to learning opportunities for teachers as well as for students. One of the teacher's roles in an inquiry classroom is to facilitate mathematical discussions. At the same time, the teacher acts as a participant who can legitimize certain aspects of the children's mathematical activity and implicitly sanction others (Lampert, 1990; Voigt, 1985). Whole-class discussions are demanding situations for teachers because they have to try to make sense of the wide array of (different) solutions offered by the children (cf. Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993). Our observations consistently indicate that teachers capitalize on the learning opportunities that arise for them as they begin to listen to their students' explanations. The increasingly sophisticated way they select tasks and respond to children's solutions, shows their own developing understanding of the students' mathematical activity and conceptual development.

These learning opportunities for the teachers are directly influenced by the sociomathematical norms negotiated in the classrooms. In particular, children continue to give a variety of explanations when different solutions are emphasized and developmentally sophisticated solutions are legitimized. These inform the teachers about the students' conceptual possibilities and their current understandings. The latter, in turn, contribute to the teachers' evolving notions of what is sophisticated and efficient for the children. This further illustrates the reflexive relationship between the establishment of sociomathematical norms and the teacher's increasing understanding of mathematical difference, sophistication, and efficiency. For a more detailed discussion of teachers' learning in inquiry mathematics classrooms see Wood, Cobb, and Yackel (1991) and Yackel, Cobb, and Wood (in press).

THE INTERACTIVE CONSTITUTION OF WHAT COUNTS AS AN ACCEPTABLE EXPLANATION AND JUSTIFICATION

We turn now to consider how the teacher and students in an inquiry mathematics classroom interactively constitute what counts as an acceptable explanation and justification and thus elaborate their taken-as-shared basis for communication. Viewed as a communicative act, explaining has as its purpose clarifying aspects of one's (mathematical) thinking that might not be apparent to others. Consequently, what is offered as an explanation is relative to the perceived expectations of others.

Our analysis of classroom data shows an evolution of students' understanding of what counts as an acceptable mathematical explanation and justification (Yackel, 1992). Initially, students' explanations may have a social rather than a mathematical basis. As their participation in inquiry mathematics instruction increases, they differentiate between various types of mathematical reasons. For example, they distinguish between explanations that describe procedures and those that describe actions on experientially real mathematical objects. Finally, some students progress to being able to take explanations as objects of reflection. In the following discussion we illustrate these three aspects of students' understanding of explanation. In each case the focus of the discussion is on the interactive constitution of what constitutes acceptability.

A Mathematical Basis for Explanations

A preliminary step in children's developing understanding of what constitutes an acceptable mathematical explanation is that they understand that the basis for their actions should be mathematical rather than status-based. Developing this preliminary understanding is not a trivial matter, especially since children are often socialized in school to rely on social cues for evaluation and on authority-based rationales. For example, in many classrooms it is appropriate for a child to infer that his answer is incorrect if the teacher questions it. In the classrooms that we have studied, one of the expectations is that children explain their solution methods to each other in small-group work and in whole-class discussions. However, most of the children were experiencing inquiry-based instruction for the first time and had little basis for knowing what types of rationales might be acceptable. In their prior experience of doing mathematics in school their teachers were typically the only members of the classroom community who gave explanations. They were therefore accustomed to relying on authority and status to develop rationales. For example, early in the school year one child attempted to resolve a dispute about an answer during small-group work by initiating a discussion about who had the best pencil and then about which of them was the smartest. This attempt to use status rather than a mathematical rationale to resolve the disagreement is consistent with the way many children interpret traditional mathematics instruction, as arbitrary procedures prescribed by their classroom authorities—the textbook and the teacher (Kamii, 1994; Voigt, 1992).

The following episode, which occurred early in the school year, demonstrates how a teacher can capitalize on situations that arise naturally in the classroom to make children's reasons an explicit topic of discussion.

Example 4: The teacher has posed a double tens-frame task using two red chips in the left tens-frame and 8 green chips in the right tens-frame.

Teacher: How many more green are there than red? How many more?
Donna: Six.
Teacher: There are six? All right. Six. Is that right class?
Students: Yes. No.
Donna: Oh, seven.
Student: Oh, I know.
Teacher: Seven.
Donna: Eight. Eight.
Student: I know. I know.
Teacher: (To Donna.) There are eight more green than there are red?
Student: No.
Student: Oh, Mr. K., I know.
Teacher: Think about it Donna. How many more green circles are there than red? Daria?
Daria: Six.
Teacher: How many?
Daria: Six.
Teacher: Is that right class? Do we agree with that?
Students: No. Yes.
Teacher: I heard some nos.

Many students begin talking at once.

Teacher: Listen. Listen.
Donna: (Protesting to the teacher) I said the six, but you said, "No."

In response to Donna's explicit acknowledgment that she changed her answers on the basis of her interpretation of the social situation rather than on mathematical reasoning, the teacher invents a scenario to clarify his expectations for this class.

Teacher: Wait, listen, listen. What did Mr. K.—what have I always taught you? (To Donna) What's your name?
Donna: My name is Donna Walters.
Teacher: What's your name?
Donna: My name is Donna Walters.
Teacher: If I were to ask you, "What's your name?" again, would you tell me your name is Mary?

Donna: No.
Teacher: Why wouldn't you?
Donna: Because my name is not Mary.
Teacher: And you know your name is—.... If you're not for sure you might have said your name is Mary. But you said Donna every time I asked you because what? You what? You know your name is what?
Donna: Donna.
Teacher: Donna. I can't make you say your name is Mary. So you should have said, "Mr. K. Six. And I can prove it to you." I've tried to teach you that.

Interventions of this type are powerful because they become paradigm cases that students can refer to. In general, such interventions are successful in establishing the expectation that rationales should be mathematical.

Explanations as Descriptions of Actions on Experientially Real Mathematical Objects

A more complex issue than establishing that mathematical reasons should form the basis for explanations, is *which types* of mathematical reasons might be acceptable. Here again, reflexivity is a key notion that guides our attempt to make sense of the classroom. We argue that what constitutes an acceptable mathematical reason is interactively constituted by the students and the teacher in the course of classroom activity. In the classroom studied, the children contributed to establishing an inquiry mathematics tradition by generating their own personally meaningful ways of solving problems instead of following procedural instructions. Further, their explanations increasingly involved describing actions on what to them were mathematical objects. In this sense, their explanations were conceptual rather than calculational (Thompson, Philipp, Thompson, & Boyd, 1994). In addition, children took seriously their obligation to try to make sense of the explanations of others. As a consequence, explanations were frequently challenged if they could be interpreted as relying on procedural instructions or if they used language that did not carry the significance of actions on taken-as-shared mathematical objects, which were experientially real for the students. These challenges in turn gave rise to situations for the teacher and students to negotiate what was acceptable as a mathematical explanation. The following illustrative episode, which occurred 2 months after the beginning of the school year, clarifies how the sociomathematical norm of what is acceptable as a mathematical explanation, is interactively constituted.

Example 5: The episode begins as Travonda is explaining her solution to the following problem.

> Roberto had 12 pennies. After his grandmother gave him some more, he had 25 pennies. How many pennies did Roberto's grandmother give him?

At Travonda's direction, the teacher writes

$$\begin{array}{r} 12 \\ \underline{+\,13} \end{array}$$

on the overhead projector. Thus far, her explanation involves specifying the details of how to write the problem using conventional vertical format. She continues.

Travonda:	I said, one plus one is two, and 3 plus 2 is 5.
Teacher:	All right, she said ...
Rick:	I know what she was talking about.
Teacher:	Three plus 2 is 5, and one plus one is two.

Travonda's explanation can be interpreted as only procedural in nature. She has not made explicit reference to the value of the quantities the numerals signify nor clarified that the results should be interpreted as 25. Furthermore, in repeating her solution, the teacher modifies it to make it conform even more closely to the standard algorithm by proceeding from right to left. Several children simultaneously challenge the explanation.

Jameel:	(Jumping from his seat and pointing to the screen.) Mr. K. That's 20. That's 20.
Rick:	(Simultaneously) Un-uh. That's 25.
Several students:	That's 25. That's 25. He's talking about that.
Jameel:	Ten. Ten. That's taking a 10 right here ... (walking up to the overhead screen and pointing to the numbers as he talks). This 10 and 10 (pointing to the ones in the tens column). That's 20 (pointing to the 2 in the 10s column).
Teacher:	Right.
Jameel:	And this is 5 more and it's 25.
Teacher:	That's right. It's 25.

Both Rick's challenge that the answer should be expressed as 25, rather than as two single digits and Jameel's challenge that the ones signify 10s and the two signifies 20 contribute to establishing the sociomathematical norm that explanations must describe actions on mathematical objects. Further, by acknowledging the challenges and accepting Jameel's clarification the teacher legitimized the ongoing negotiation of what is acceptable as an explanation in this classroom.

As a communicative act, explanation assumes a taken-as-shared stance (Rommetveit, 1985). Consequently, what constitutes an acceptable explanation is constrained by what the speaker and the listeners take as shared. But, as the above example shows, what is taken as shared is itself established during class discussions. Further, our analyses of discussions across the school year document that what is taken-as-shared mathematically evolves as the year progresses. Here, Jameel's clarification assumes that the conceptual acts of decomposing 12 into 10 and 2 and of decomposing 13 into 10 and 3 are shared by other students. Individual interviews conducted with all of the children in the class shortly before this episode occurred indicate that for a number of students this was not the case. Thus, although Jameel's explanation made it possible for him to orient his own understanding to Travonda's reported activity, it may have been inadequate for others.

Explanations as Objects of Reflection

When students begin to consider the adequacy of an explanation for others rather than simply for themselves, the explanation itself becomes the explicit object of discourse (Feldman, 1987). During classroom discussions, it is typically the teacher's responsibility to make implicit judgments about the extent to which students take something as shared and to facilitate communication by explicating the need for further explanation. As students' understanding of an acceptable explanation evolves, they too may assume this role. To do so, they must go beyond making sense of an explanation for themselves to making judgments about how other children might make sense of it. This involves a shift from participating in explanation to making the explanation itself an object of reflection. This shift in students' thinking is analogous to the shift between process and object that Sfard (1991) describes for mathematical conceptions. In the same way that being able to see a mathematical entity as an object as well as a process indicates a deeper understanding of the mathematical entity, taking an explanation as an object of reflection indicates a deeper understanding of what constitutes explanation.

The following example clarifies the shift in thinking that accompanies focusing on the explanation itself as an object. The episode occurred close to the end of the school year.

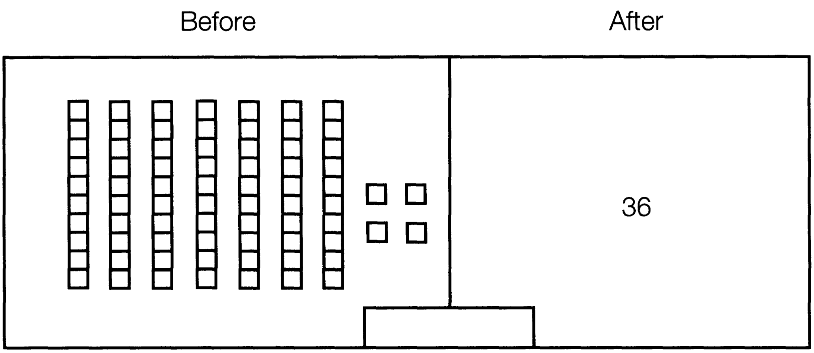

Figure 2. Problem task as shown on student activity page.

Example 6: Daria and Donna use centicubes on the overhead projector to explain their solution to the problem shown in Figure 2.

The task is to figure out how much to add to or subtract from what is shown "before" to get what is shown "after." The girls had arrived at 38 as an answer during small-group work. To describe their solution to the class, they first place 74 centicubes on the overhead projector, using seven strips of ten (strips) and four individual cubes (squares).

Daria: We took this 40 off (points to four strips which the teacher then removes). That left 34. Oh, (to the teacher) put a 10 back. (The teacher replaces one of the strips.) 35, 36 (pointing to two of the cubes in the additional strip).

In our experience, purely conceptual solutions to tasks of this type require part-whole reasoning with tens and ones. This appears to be beyond the current conceptual capabilities of many second graders, and they needed to use manipulative or visual materials both to solve the tasks and to understand others' explanations. However, the strip (of 10 ones) the girls pointed to when they said, "35, 36" appeared as a single object on the overhead screen. Only those children who were looking directly at the materials laid on the overhead projector could see the 10 ones that composed the strip. The visual material available to the girls giving the explanation and to the children listening to the explanation, except for those children sitting immediately next to the overhead projector, was not the same. This subtle, but significant, point is indicated by Jameel's question.

Jameel: How—Wait, I got a question.

Teacher: Wait a minute, count that—

Jameel: Hey, Mr. K. If—How could she know, if you show two—How could the other person see if she does like when she said 44, 45, 46? How could she know it was two strips, I mean how could they know it was two squares like that? (Jameel appears to misspeak when he says 44, 45, 46 instead of 34, 35, 36.)

Toni: 'Cause they can see it.

Rick: No, we can't. We can't see it.

Jameel's question initiates a shift in the discussion from the solution of the problem to the adequacy and clarity of the explanation. At first glance, it may seem that his challenge is simply about the use of the manipulative materials. However, Toni's and Rick's responses and the subsequent discussion clarify that the issue is the coordination of tens and ones. Toni's reaction is interesting, given what we know about her conceptual possibilities. She is one of the children who would need to have manipulative or visual materials to solve the problem. However, she, like Jameel, was sitting immediately next to the overhead projector, and she looked at what Daria was actually pointing to rather than at what was visible on the overhead screen. Rick, however, is one of the children who would be able to solve the problem without using manipulatives. His "No, we can't. We can't see it," indicates that he shares Jameel's understanding that Daria's explanation has not clarified that the strip can be thought of as 10 ones.

The episode continues when the girls ask if there are any other questions. Jameel insists that the explanation requires elaboration, and the girls explain their solution again. Now, Daria actually removes 38 cubes in an attempt to demonstrate their solution. She removes three strips and the four individual cubes and breaks four additional cubes off of one of the remaining strips, leaving six connected cubes.

Students: Take those (strip of six) apart.

Teacher: Take those apart.

The girls break the six connected cubes apart, making it possible for all of the children to see them individually and therefore to count them. Finally, Daria counts to verify that there are 36, pointing as she counts, "10, 20, 30, 31, 32, 33, 34, 35, 36." This final explanation provides the explication that Jameel called for.

The preceding episode is significant because it shows that at least some of the children went beyond trying to make sense of an explanation for themselves and considered the extent to which it might be comprehensible to other members of the class. Jameel's criticism of the explanation was not that it didn't make sense to him. Rather, it was that those who could not see the 10 ones in the 10-strip might not be able to make sense of it. Jameel's question shifted the focus of the discussion from the solution of the problem to the adequacy of the explanation. In doing so, he made the explanation itself an object of reflection for others in the class as well as for himself.

INTELLECTUAL AUTONOMY

The development of intellectual and social autonomy is a major goal in the current educational reform movement, more generally, and in the reform movement in mathematics education, in particular (National Council of Teachers of Mathematics, 1989). In this regard, the reform is in agreement with Piaget (1948/1973) that the main purpose of education is autonomy. Prior analysis shows that one of the benefits of establishing the social norms implicit in the inquiry approach to mathematics instruction is that they foster children's development of *social autonomy* (Cobb, et al., 1991; Cobb, Yackel, & Wood, 1989; Kamii, 1985; Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990). However, it is the analysis of *sociomathematical norms* implicit in the inquiry mathematics tradition that clarifies the process by which teachers foster the development of intellectual autonomy.

In this account, the conception of autonomy as a context-free characteristic of the individual is rejected. Instead, autonomy is defined with respect to students' participation in the practices of the classroom community. In particular, students who are intellectually autonomous in mathematics are aware of, and draw on, their own intellectual capabilities when making mathematical decisions and judgments as they participate in these practices (Kamii, 1985). These students can be contrasted with those who are intellectually heteronomous and who rely on the pronouncements of an authority to know how to act appropriately. The link between the growth of intellectual autonomy and the development of an inquiry mathematics tradition becomes apparent when we note that, in such a classroom, the teacher guides the development of a community of validators and thus encourages the devolution of responsibility. However, students can take over some of the traditional teacher's responsibilities only to the extent that they have constructed personal ways of judging that enable them to know in action both when it is appropriate to make a mathematical contribution and what constitutes an acceptable mathematical contribution. This requires, among other things, that students can judge what counts as a different solution, an insightful solution, an efficient solution, and an acceptable explanation. But, as we have attempted to illustrate throughout this paper, these are the types of judgments that the teacher and students negotiate when establishing sociomathematical norms that characterize an inquiry mathematics tradition. In the process, students construct specifically mathematical beliefs and values that help form their judgments. For instance, Jameel's challenge that "one and one is two" signifies "ten and ten is twenty" illustrates that children are capable of making judgments about what is appropriate mathematically. Further, Jameel's challenge indicates that he had developed the belief that mathematical explanations should describe actions on experientially real mathematical objects. Examples such as this show that it is precisely because children can make personal judgments of this kind on the basis of their mathematical beliefs and values that they can participate as increasingly autonomous members of an inquiry mathematics community.

SIGNIFICANCE

The notion of sociomathematical norms that we have advanced in this paper is important because it sets forth a way of analyzing and talking about the *mathematical* aspects of teachers' and students' activity in the mathematics classroom. This is a significant extension of prior work on general classroom social norms in that it clarifies aspects of teachers' and students' activity that sustain a classroom atmosphere conducive to problem solving and inquiry. These sociomathematical norms are intrinsic aspects of the classroom's mathematical microculture. Nevertheless, although they are specific to mathematics, they cut across areas of mathematical content by dealing with mathematical qualities of solutions, such as their similarities and differences, sophistication, and efficiency. Additionally, they encompass ways of judging what counts as an acceptable mathematical explanation.

We have also attempted to demonstrate that these norms are not predetermined criteria introduced into the classroom from the outside. Instead, these normative understandings are continually regenerated and modified by the students and the teacher through their ongoing interactions. As teachers gain experience with an inquiry approach to mathematics instruction they may have some clear ideas in advance of norms that they might wish to foster. Even in such cases these norms are, of necessity, interactively constituted by each classroom community. Consequently, the sociomathematical norms that are constituted might differ substantially from one classroom to another. For purposes of this paper, we have discussed the development of sociomathematical norms in classrooms that generally follow an inquiry form of instruction. As we have shown, in the process of negotiating sociomathematical norms, students in these classrooms actively constructed personal beliefs and values that enabled them to be increasingly autonomous in mathematics.

The notion of sociomathematical norms is also important for clarifying the teacher's role as a representative of the mathematical community. The question of the teacher's role in classrooms that attempt to develop a practice consistent with the current reform emphasis on problem solving and inquiry is one of current debate (Clement, 1991). Many teachers assume that they are expected to assume a passive role (P. Human, personal communication, August 1994). However, we question this position. As we have stated previously,

> The conclusion that teachers should not attempt to influence students' constructive efforts seems indefensible, given our contention that mathematics can be viewed as a social practice or a community project. From our perspective, the suggestion that students can be left to their own devices to construct the mathematical ways of knowing compatible with those of wider society is a contradiction in terms. (Cobb, Yackel, & Wood, 1992, pp. 27–28)

In this paper we have attempted to clarify one critical aspect of the teacher's role in influencing the mathematical aspects of the knowledge children construct. In this regard, the ideas set forth in this paper are potentially useful in preservice and inservice teacher education. For example, in a recent project classroom teaching experiment, the notion of sociomathematical norms influenced discussions between the researcher and the classroom teacher. In particular, the issue of what constitutes a mathematically efficient solution became an explicit focus in discussions with the teacher and in the classroom itself. In the process, the level of discourse and the individual children's learning advanced (Cobb, Boufi, McClain, & Whitenack, in press).

The analysis of sociomathematical norms indicates that the teacher plays a central role in establishing the mathematical quality of the classroom environment and in establishing norms for mathematical aspects of students' activity. It further highlights the significance of the teacher's own personal mathematical beliefs and values and their own mathematical knowledge and understanding. In this way, the critical and central role of the teacher as a representative of the mathematical community is underscored.

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