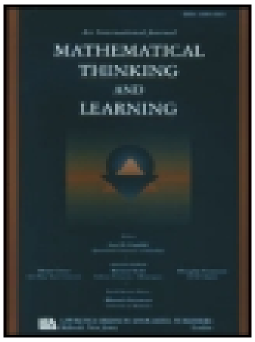

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ARTICLES

A Comparative Analysis of the Addition and Subtraction of Fractions in Textbooks from Three Countries

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In this paper, we report on a comparison of the treatment of addition and subtraction of fractions in primary mathematics textbooks used in Cyprus, Ireland, and Taiwan. To this end, we use a framework specifically developed to investigate the learning opportunities afforded by the textbooks, particularly with respect to the *presentation* of the content and the textbook *expectations* as manifested in the associated tasks. We found several similarities and differences among the textbooks regarding the topics included and their sequencing, the constructs of fractions, the worked examples, the cognitive demands of the tasks, and the types of responses required of students. The findings emphasized the need to examine textbooks in order to understand differences in instruction and achievement across countries. Indeed, we postulate that within a given country there may exist a recognizable "textbook signature." We also draw on the results and the challenges inherent in our analysis to provide suggestions and directions for future textbook analysis studies.

This paper is an application of a synthesis of research conducted by the first three authors while taking a class on curriculum in mathematics education taught by the last author. We opt for an alphabetical listing of names, but contributions were at comparable levels. Preliminary reports on this work were presented at the annual meeting of the *American Educational Research Association*, in Chicago, 2007 and the 31st meeting of the International Group of the *Psychology of Mathematics Education*, in Seoul, 2007. We thank the editor and three anonymous reviewers for their comments on an earlier version of this paper and the publishers of the books, who granted us permission to reproduce the illustrative excerpts in this article.

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Researchers use different ways to study the important question of how mathematics textbooks in different countries structure learning opportunities for their students. No agreed approach has yet emerged for evaluating and comparing learning opportunities in textbooks. Furthermore, researchers have expressed contrasting views about what can be learned from analyzing mathematics textbooks. Some researchers claim that textbook analysis can explain differences in students' performance in international comparative studies (Fuson, Stigler, & Bartsch, 1988; Li, 2000). Other researchers, however, have argued that textbooks exert little influence on instruction and on what students learn (Freeman & Porter, 1989). Yet other researchers have viewed textbooks as a potential source for teacher learning—a goal that is frequently unfulfilled (Newton & Newton, 2006; Remillard, 2005).

A more moderate viewpoint posits that textbooks afford *probabilistic* rather than deterministic opportunities to learn mathematics (Mesa, 2004; Valverde, Bianchi, Wolfe, Schmidt, & Houang 2002). Viewed this way, textbook analysis is what Mesa (2004) has termed

> a hypothetical enterprise [which asks]: What *would* students learn if their mathematics classes were to cover all the textbook sections in the order given? What *would* students learn if they had to solve all the exercises in the textbook? (pp. 255–256)

The conditional tense of these two questions reflects the challenge faced by textbook reviewers because the role of textbooks in instruction depends on how students and teachers interact with them in instruction. Although this perspective acknowledges that textbook analysis examines only the *intended curriculum* (the goals and objectives for mathematics intended for learning at a national or regional level[1]) and not the implemented curriculum, studying textbooks used in different countries can reveal similarities and differences in opportunities to learn mathematics[2] offered to students around the world. For example, such analyses can reveal different performance expectations made of students in different countries, the extent to which a country's textbook series prioritizes conceptual understanding or procedural fluency, and how the treatment of the mathematical content differs among countries (e.g., Li, 2000).

In this paper, we take one primary-school topic—the addition and subtraction of fractions—and illustrate how a framework we specifically developed to attend to both general and particular aspects of the textbook can be used to characterize learning opportunities in textbooks from three countries: Cyprus, Ireland, and Taiwan. We chose to analyze the addition and subtraction of fractions because of the topic's ubiquity on primary school curricula around the world (Mullis, Martin, Gonzalez, & Chrostowski, 2004), the availability of research on it,[3] and because operations with fractions, in general, is a difficult topic to learn and to teach (Lamon, 1999). We use the topic as a case to study how textbooks from three countries treat one topic, thus taking on Thompson and Saldanha's (2001) challenge to understand how "countries' cultures and educational systems support students' creation of . . . fractions" (p. 109).

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[1]Or the publisher's interpretation of those goals and objectives.

[2]For the purposes of this paper, *opportunity to learn* refers to a description of the complexity of the mathematics that is present in the textbooks that students would use in their classrooms, thus limiting its definition to the mathematical content as opposed to the more general notion that includes school factors that most directly affect student learning. For an historical description of the evolution of this notion, see McDonnell (1995) and Tate and Rousseau (2006).

[3]Interestingly, however, the topic has been studied less extensively than multiplication or division of fractions (Verschaffel, Greer, & Torbeyns, 2006, p. 65).

Using our framework for in-depth study of one topic in textbooks from different countries would highlight specific differences and similarities in textbooks, which may be indicative of wider (cultural) similarities and differences. The three countries, Cyprus, Ireland, and Taiwan, were selected because each has a national curriculum, and thus approved textbooks are likely to reflect the curriculum that public school students in each country are required to study. Moreover, the three countries differ significantly in history, size, language, economy, culture, and student attainment in international comparative studies. We anticipated that these differences would be reflected in how opportunities to learn are expressed in the mathematics textbooks. In addition, our combined experiences as students and teachers of primary mathematics in these countries provided us with country-specific knowledge conducive to such an analysis.

Based on our chosen topic and countries, we identified two research questions:

1. What similarities and differences can be observed in the presentation of addition and subtraction of fractions in primary mathematics textbooks in Cyprus, Ireland, and Taiwan?
2. What expectations of student performance on addition and subtraction of fractions are embedded in primary textbook tasks in Cyprus, Ireland, and Taiwan?

In what follows, we first summarize relevant literature and identify the methods used to respond to the two research questions; we then report our findings, discuss their implications for researchers interested in textbook analysis, and conclude by outlining further research steps.

LITERATURE REVIEW

In this section, we summarize literature on comparative analysis of textbooks, which informed our analysis and, particularly, the framework that we developed to this end. We then briefly review literature on students' difficulties with our chosen topic—addition and subtraction of fractions.

Textbook Analysis in Cross-National Studies

Studies have identified significant cross-national differences among mathematics textbooks. Two large-scale studies that compared and contrasted the textbooks used in almost 40 countries concluded that textbooks vary in many ways (Schmidt, McKnight, Valverde, Houang, & Wiley, 1997) and that "they exhibit substantial differences in presenting and structuring pedagogical situations" (Valverde et al., 2002, p. 17). Small-scale studies also found major differences between mathematics textbooks used in China, Japan, the former Soviet Union, Taiwan, and the United States (e.g., Fuson et al., 1988; Mayer, Sims, & Tajika, 1995; Stevenson & Bartsch, 1992), and among the textbooks used in different European countries (Pepin, Haggarty, & Keynes, 2001).

In general, we found three broad categories of approaches to cross-national textbook-analysis that we classified as *horizontal, vertical,* and *contextual.* In the horizontal analysis the textbook is examined as *a whole,* as a piece of technology in the educational system (Herbst, 1995) and the analysis focuses on general textbook characteristics (e.g., physical appearance, the organization of the content across the book, Schmidt et al., 1997; Stevenson & Bartsch, 1992). Critics have argued that this approach overlooks fundamental differences in the learning opportunities offered to students in different countries, because different textbooks do not treat topics in the same manner and with the same degree of emphasis (Howson, 1995). For example, Stevenson and Bartsch (1992) compared the content of Japanese and American K-12 textbooks, identifying which topics appeared in each country and in which grade level. Their study showed that some concepts were introduced earlier in the American textbooks and others earlier in the Japanese textbooks; also some of the concepts introduced in the textbooks of one country were not introduced in the textbooks of the other country. Although their findings provided preliminary insights into the treatment of the content in the textbooks of these countries, they left other questions unanswered, key among them being how concepts were treated within each textbook to structure learning opportunities for students.

The vertical analysis examines how textbooks treat a *single* mathematical concept (Li, 2000; Mesa, 2010) and views the textbook as an "environment for construction of knowledge" (Herbst, 1995, p. 3).[4] Such approaches overlook how the treatment of the topic being examined relates to other topics contained in the textbook. The third approach, *contextual analysis*, attends to the ways in which textbooks are used in instructional activities by either instructors or students (Mesa, 2007; Remillard, 2005; Rezat, 2006) and sees the textbook as an "artifact in the broader sense . . . historically developed, culturally formed, produced for certain ends, and used with particular intentions" (Rezat, 2006, p. 409). The third approach deals directly with problems of implementing the curriculum; that is, of realizing the intentions of the textbooks. Because we are interested in understanding the *intentions* of the textbook authors and publishers and how other researchers have proceeded to analyze such intentions, we did not include the contextual approach in our discussion. We believe, however, that a framework that incorporates both horizontal and vertical analyses of textbooks is a prerequisite for a contextual analysis of textbooks. Combining both dimensions of analysis could reveal characteristics of textbooks that would be lost in analyzing only one of the dimensions.

Our review of the literature has yielded at least two studies that have pursued both horizontal and vertical textbook analyses (Haggarty & Pepin, 2002; Howson, 1995); both these studies illustrate that such a complementary approach is not only feasible but also worthwhile because it provides a means for better exploring the opportunities to learn that students (and instructors) are afforded as they engage with mathematics textbooks. However, the lack of common and explicit criteria for textbook comparisons across both studies makes it difficult for readers to make meaningful comparisons and to replicate the analyses undertaken in those studies. Hence, we saw a need to develop a framework that integrates vertical and horizontal analyses and is explicit about what is being analyzed, facilitating consistency of comparisons across textbooks and countries. Because the development and the application of this framework constitute a significant part of our comparative analysis, this framework is briefly detailed in the Methods section.

Addition and Subtraction of Fractions

Students' difficulties in adding and subtracting fractions have concerned educators for years. For instance, in the early 1930s, the *Committee of Seven* in the United States launched a study to

[4]In our literature search, we found this type of textbook analysis frequently, not only in mathematics but in other disciplines as well. For example, Beck, McKeown, and Gromoll (1989) analyzed the presentation of specific topics in four series of social study textbooks for fourth and fifth grade.

examine the grade at which the addition and subtraction of fractions should be placed in order to facilitate students' mastery in this area (Raths, 1932). Researchers have also explored what causes student errors such as adding (or subtracting) the numerators and the denominators. This exploration suggested that several cognitive factors might explain such errors. For instance, it was proposed that students often view fractions as two separate whole numbers (one corresponding to the numerator and another to the denominator) rather than as individual quantities (Carpenter, Coburn, Reys, & Wilson, 1976), and that they are often misguided by inappropriate analogies (e.g., they believe that when adding or subtracting fractions it is legitimate to add or subtract the numerators and denominators because in the multiplication of fractions one multiplies the numerators and the denominators; Vinner, Hershkowitz, & Bruckheimer, 1981).

Students' difficulties in adding and subtracting fractions cannot be separated from their difficulties in learning fractions generally. Studies suggest that such difficulties have their roots in the complexity of the notion of fractions and in instructional approaches employed when teaching fractions (Ball, 1993; Behr, Harel, Post, & Lesh, 1993; Lamon, 1999). For example, meanings, models, and symbols that worked well for students when working on whole numbers may interfere with students' developing understanding of fractions (Lamon, 1999). Furthermore, fractions make up a multifaceted concept, consisting of five interrelated constructs: part-whole, measure, operator, quotient, and ratio (Kieren, 1976). Because each construct captures different aspects of fractions, constructing a comprehensive schema of fractions requires developing a robust understanding of all five constructs and of their confluence (Behr et al., 1993).

Research has also shown that instruction may impede the learning of fractions, especially when (1) it fails to build on students' prior knowledge (Mack, 2001), (2) it emphasizes rote learning at the expense of conceptual understanding (Ball, 1993; Mack, 2001), (3) it introduces formal symbols and algorithms before familiarizing students with the different aspects of the notion of fractions (Smith, 2002), or (4) it emphasizes only one of the constructs (usually the part-whole construct, Moss & Case, 1999). Textbooks, as one tool of instruction, may contribute to compounding or ameliorating such difficulties. In the framework used in the present study such criteria are given detailed consideration.

METHODS

Although our focus in this paper is on the comparison of the treatment of addition and subtraction of fractions in textbooks from different countries, in this section we first document the development of the framework employed in this study and then discuss the criteria that guided our inquiry. We do so because, as explained earlier, the process of developing this framework was an integral part of our approach; additionally, the results of our analysis are inextricably linked to the framework that guided our analysis. After briefly detailing the development of the framework and the particular criteria that informed our analysis, we provide background information of the educational systems under consideration and describe the data collection and analysis processes.

Framework Development

The framework used in this study (Charalambous, Delaney, Hsu, & Mesa, 2007; Delaney, Charalambous, Hsu, & Mesa, 2007) was developed in three stages. We started with a comprehensive review of the literature and used this review to identify commonalities and gaps in existing frameworks. Next we selected criteria used in these studies to characterize textbooks; finally we organized the criteria into categories and subcategories.

Our literature review suggested two issues. First, textbook features such as page size, number of pages, topics, and topic sequences (i.e., using a horizontal approach) give readers an initial glimpse of whether the textbook is appropriate for students at a given grade level or whether topic sequence, development, and relative emphasis matches a teacher's requirements. Yet, such features provide little detailed information about how authors deal with the content. A vertical analysis, on the other hand, offers more focused and in-depth analyses of the mathematical content. A framework that incorporates both dimensions benefits from the strengths of each. Second, within each dimension of analysis different aspects of the textbooks can be examined (e.g., mathematical, pedagogical, sociocultural, cf. Pepin et al., 2001). Therefore, we sought to develop a framework that analyzes textbooks at two dimensions (vertical and horizontal) and that also attends to different textbook aspects.

But any attempt to develop such a framework runs the risk of being a compilation of disparate criteria with no underlying structure or organization. This could limit the framework's potential as a tool for capturing and understanding different learning opportunities that textbooks offer students (and teachers). Thus, we organized the criteria into categories and subcategories. For the horizontal-analysis dimension, we included two categories: *background information* and *overall structure* of the textbook. The first category would provide a descriptive overview of a textbook and the background of its production; the second pertained to the topics covered in the textbook and their organization. For the vertical-analysis dimension, we grouped criteria into three categories: the first category, *Communicated to students*, refers to how the textbook conveys the mathematics to students; the second category, *Required of students*, refers to the requirements that the textbook has for students; and the third category, *Connections*, analyzes explicit connections made among mathematical topics, between the textbook and other classroom work, and to situations outside school (see Figure 1). Considering that learning mathematics might encompass both mastering the content of a topic, and developing certain skills and attitudes, the first category included three subcategories of communication: *mathematical content*, *mathematical practices*, and *attitudes* toward the subject. The second category included two criteria of textbook requirements: *potential cognitive demands* and *types of responses*, while the third category included three criteria of connections: *connections within and between strands, connections between classroom instruction and textbooks, and connections to situations outside of school*.

Once the framework was developed, we explored its sensitivity to characteristics of the textbooks from the three countries. This led to elaborations of some of the subcategories and corresponding criteria, to account for such differences. For instance, in one country worked examples were typically fully worked out whereas in a second country, worked examples did not provide final answers, challenging our definition of a worked example. In resolving such ambiguities we maintained the integrity of the mathematics by consulting a mathematician and relevant literature.

The Criteria Employed in the Present Study

For the purposes of this paper, we focused on six criteria (two from the horizontal and four from the vertical dimension). Because our intention here is to explore similarities and differences in

HORIZONTAL ANALYSIS OF THE TEXTBOOK

Background Information
- Title
- Number of books
- Pages (Number and Density)
- Profile of authors and advisory committee
- Publisher and year of publication
- Accompanying materials (e.g., teachers' guides, resource materials)

Overall Structure
- Number of units/lessons and average number of pages per unit/lesson
- Structure of units/lessons
- Topics covered
- Sequencing of topics

VERTICAL ANALYSIS OF THE TEXTBOOK

Communicated to Students	**Required of Students**	**Connections**
Mathematical Content • Topic-specific construct, structure etc. (e.g. part-whole, ratio, operator, quotient, measure fraction constructs) • Definitions, rules, conventions • Illustrations-representations (irrelevant, relevant to the context but not to the mathematics, supporting the mathematics) *Mathematical Practices* • Worked examples • Modeling thinking *Attitudes* • Equity • View of mathematics	• Potential Cognitive Demands (memorization, procedures with connections, procedures without connections, doing mathematics) • Type of Response (answer only, answer and mathematical sentence, explanation, justification)	• Connecting within and between strands • Classroom instruction - textbook connections • Connecting to situations outside of school

Key: Dimension: Uppercase letters; Categories: bold; Sub-categories: italicized; Criteria: bulleted points.

FIGURE 1 The framework used to analyze the mathematics textbooks.

the treatment of the addition and subtraction of fractions among the three countries, we selected a variety of categories, subcategories, and criteria to illustrate different aspects of the potential of textbooks to support student learning. From the horizontal analysis, we considered the topics covered with respect to the addition and subtraction of fractions and their sequencing. From the *Communicated to Students* category of the vertical analysis, we considered the fraction constructs (i.e., part-whole, ratio, operator, quotient, and measure) and the worked examples used in the textbooks. From the *Required of Students* category, we focused on the potential cognitive demands of textbook tasks and the type of response asked for.

Examining the topics included in the textbooks was necessary, because a topic is more likely to be taught if it is included in the textbook than if it is omitted—although the presence of a topic does not ensure that it is taught (Valverde et al., 2002). We considered topic sequence because how different topics are ordered in textbooks might reflect cultural differences as to how mathematical thinking is perceived to develop (Schmidt et al., 1997).

We examined different fraction constructs in textbooks because previous studies suggested that different constructs can help students understand different facets of the notion of fraction (Lamon, 1999). Moreover, constructs differ in their potential to support students' competence in adding and subtracting fractions (Charalambous & Pitta-Pantazi, 2007). We analyzed worked examples because they are important means of scaffolding student understanding and can facilitate skill acquisition (Renkl, 2002). Finally, we analyzed the potential cognitive demands of tasks and the type of answers required of students. A task's cognitive complexity can determine the possibilities for student engagement in cognitively challenging tasks during instruction (cf. Stein, Grover, & Henningsen, 1996); when students explain and justify their answers, their understanding of mathematics is reinforced (NCTM, 2000).

The Setting of the Study: Educational Systems and Textbooks

Notable similarities and differences in the education systems of the three countries justified their suitability for a cross-national comparison (see Table 1). Each country has a centralized educational system and a national curriculum or national standards. Primary schooling, which is compulsory in all three countries, covers grades 1 to 6 in Taiwan and Cyprus and Pre-K to 6 in Ireland. The three countries differ, however, in the school time allotted to mathematics. Whereas the academic year in Cyprus and Ireland is about 36 weeks, in Taiwan it is somewhat longer (approximately 40 weeks). Cypriot students, in contrast, spend more time on mathematics every week compared to their counterparts in Ireland and Taiwan (240 minutes in Cyprus compared to 180 minutes in Ireland and to a range of 100–150 minutes in Taiwan). But in these two countries, teachers have extra minutes that can be allocated to any subject, including mathematics, at the discretion of the individual school or teacher (120 minutes in Ireland and a range of 120–240 minutes in Taiwan).

Differences were also identified in the publication and distribution of the mathematics textbooks.[5] In Cyprus only one textbook series is available, and it is published and distributed at no cost to students by the Ministry of Education and Culture. Thus, all public primary schools use the same mathematics textbook series. In Ireland and Taiwan textbooks are published by private publishing companies; in Taiwan, however, the Ministry of Education approves textbooks based on whether they meet the national standards. Irish and Taiwanese students purchase their textbooks, although students whose parents cannot afford to pay for them are supported financially. Because multiple textbooks series are available in Ireland and Taiwan, it was possible to examine whether there is greater variation of the textbook characteristics among rather than within different countries.

Table 1 shows that despite their differences, the three countries have comparable goals for educating their students mathematically: all three emphasize problem solving, reasoning, communication, building of connections, and application of mathematics to real-life

[5]Our discussion of the textbooks used in the three countries is grounded in a set of criteria from the "Background Information" category of the *Horizontal Analysis* dimension of the framework presented in Figure 1 (i.e., title of the textbooks, number of books, number of pages per book, publisher and year of publication, and accompanying materials).

TABLE 1
Background Information

	Cyprus	*Ireland*	*Taiwan*
Educational system	Centralized	Centralized	Centralized
Primary education	Compulsory (Grades 1–6)	Compulsory (Grades PreK–6)	Compulsory (Grades 1–6)
Academic year	Approximately 36 weeks	Approximately 36 weeks	Approximately 40 weeks
Time allotted to mathematics (weekly)	240 minutes	180 minutes (120 additional minutes may be allocated to different subjects, including mathematics, at each teacher's or school's discretion)	Ranges from 100–150 minutes (120–240 additional minutes to be allocated to different subjects, including mathematics, at school's discretion)
Curriculum	National	National	National
Available textbooks	One; published by the Ministry of Education and Culture	Multiple; published by publishing companies	Multiple; published by publishing companies
Price	Given for free	Bought by parents (financial support available for less affluent students)	Bought by parents (financial support available for less affluent students)
Curriculum emphases	Problem solving; Reasoning; Communication; Connections; Applications to real-life situations; Collaboration/group work	Problem solving; Reasoning; Communication; Connections; Applications to real-life situations; Quantitative and spatial thinking	Problem solving; Reasoning; Communication; Connections; Applications to real-life situations; Quantitative thinking; Mathematical attitude
Available curriculum materials	Student books (four volumes) One teacher manual One assessment textbook	One student book One teacher manual Posters and work cards Assessment textbook	Two student textbooks Two student workbooks One student manipulative book Two teacher manuals
Textbooks analyzed in this study	*Mathematics for Fourth Grade* (Ministry of Educational and Culture, 1998) [CT]	*Mathemagic 5* (Barry et al., 2003) [ITA] *Action Maths: 5th Class* (Courtney, 2002) [ITB]	*Fourth-Grade Mathematics* (Mou & Yang, 2004, 2005) [TTA] *Fourth-Grade Mathematics* (Li, 2004, 2005) [TTB]
Total number of pages; page size*	468 pages; 20cm x 20cm	ITA: 186 pages; 21cm x 27cm ITB: 188 pages; 21cm x 28cm	TTA: 353 pages; 19cm x 26cm, TTB: 413 pages; 19cm x 26cm

*The number of pages and the paper size is not very informative because the page density differed significantly across the textbooks of the three countries.

situations. In addition, the Irish curriculum emphasizes quantitative and spatial thinking, the Taiwanese curriculum aims to support students' quantitative thinking and development of positive mathematical attitude, and the Cypriot one emphasizes the value of collaboration (Government of Ireland, 1999; Ministry of Education, 2000; Ministry of Education and Culture, 1998).

The Cypriot fourth-grade textbook (Ministry of Education and Culture, 1998)—hereafter referred to as CT—consists of four volumes (totaling 468 pages), a teacher guide, and summative tests for each textbook unit. Because the textbook follows a spiral approach in its presentation of the content, fractions are interspersed over seven of the eight CT chapters (occupying 15% of the textbook pages). Addition and subtraction of fractions occupies 18 pages (4% of the CT pages) and mostly appears in consecutive pages in Chapter 5 (pp. 44–48), Chapter 6 (pp. 110–112, 126), Chapter 7 (pp. 44, 46, 48–52), and Chapter 8 (pp. 102–103). While Chapters 5 and 6 deal with the addition and subtraction of proper fractions, Chapters 7 and 8 also focus on the addition and subtraction of mixed numbers.

The two Irish textbooks analyzed (Barry, Manning, O'Neill, & Roche, 2003-hereafter referred to as ITA; and Courtney, 2002—ITB) include a student book (186 pages for ITA and 188 for ITB) and a teacher manual. In ITA, fractions occur in two chapters (Chapters 8 and 11 out of 45 chapters) of the book; there are also two additional chapters on fractions, decimals, and percentages. All the work on addition and subtraction of fractions is contained in Chapter 11 and covers four consecutive pages (pp. 47–50). These pages take up 2% of the pages of ITA. In ITB, fractions occur in two chapters (Chapters 7 and 12). All the work on addition and subtraction is contained in 4.5 consecutive pages of Chapter 7 (pp. 41–51). Chapter 12 (pp. 86–92) refers only to addition of fractions in the context of introducing multiplication of fractions. Addition and subtraction of fractions in ITB occupies 2% of the textbook, which includes 25 chapters. In both Irish textbooks occasional questions on addition and subtraction of fractions appear in subsequent revision chapters.

The Taiwanese textbooks analyzed (Mou & Yang, 2004, 2005—hereafter referred to as TTA; and Li, 2004, 2005—TTB) include student textbooks and student workbooks (two volumes each, 353 pages for TTA and 413 for TTB), as well as student manipulative books and teacher guides. In TTA, which consists of 20 chapters, fractions appear in two chapters (Chapter 5 in Volume 1 and Chapter 9 in Volume 2) and in an appendix in Volume 2. All the pages of Chapter 9 (pp. 80–87 of student textbook and pp. 68–73 of student workbook) are allotted to the addition and subtraction of fractions. These pages take up 4% of the TTA pages. In TTB, which contains 21 chapters, fractions occur in three chapters (Chapter 2 and Chapter 7 in Volume 1 and Chapter 2 in Volume 2) and in an appendix in Volume 2. Addition and subtraction of fractions occupies 31 pages (8% of the TTB pages) and mostly appears on the following pages: Chapter 2, Volume 1 (pp. 12, 15–18 of student textbook and pp. 13–14, 18 of student workbook); Chapter 7, Volume 1 (pp. 68–69, 76–79 of student textbook and pp. 49, 53–57 of student workbook); Chapter 2, Volume 2 (pp. 22–25 of student textbook and pp. 22–25 of student workbook); and the appendix (pp. 118–119 of student textbook). In both TTA and TTB occasional questions on addition and subtraction of fractions appear in subsequent review chapters.

Because we focused on the written curriculum (i.e., the objectively given structure of the textbook) and not on how the curriculum is enacted during instruction (i.e., the subjective scheme, cf. Herbel-Eisenmann, 2007), we analyzed only the student textbooks; this included the Taiwanese workbooks, which contain the exercises that students are expected to do. Exploring teacher guides and other ancillary materials in addition to student textbooks might provide a more comprehensive picture of how addition and subtraction of fractions are treated in textbooks from the three countries. But such an exploration was beyond the scope of the present study.

Data Collection and Analysis

Besides selecting the topic of fraction addition and subtraction, two additional sampling decisions framed our analysis: selecting grade levels and selecting textbooks on which to focus our analysis.

Selecting the grade levels on which to focus our analysis was a challenge. A direct comparison of textbooks across grades in the three countries would be problematic for several reasons. First, different countries allocate diverse amounts of time to mathematics instruction. Ireland's curriculum allocates only three hours per week to mathematics in classes from first grade to sixth, whereas Cyprus allocates four hours. Taiwan allocates less time per week on mathematics instruction in fourth grade than Ireland does, but students in Taiwan attend school for almost 20 additional days per year. Similarly, students' ages may vary across grades because Irish students spend eight years in primary school from the age of four or five whereas students in Cyprus and Taiwan spend six years from about age six.

Second, based on content included in earlier grade levels, textbooks in different countries may exhibit different assumptions about the prior knowledge students are assumed to possess when a topic is introduced in a particular grade. For example, in Taiwan students are introduced to "composition and decomposition" of fractions in third grade; they first compose and decompose fractions with concrete materials (e.g., pizza) and then connect the activities using concrete materials with algorithms for adding and subtracting fractions. The introduction of this topic in third grade might influence how it is developed in the textbook at the next grade level.

A third problem in selecting grade levels was that in Taiwan the curriculum was reformed twice between 1999 and 2006, leading to changes in how topics were organized in textbooks over this period. In Ireland, the curriculum was revised in 1999 and has not been revised since; textbooks produced in 2002 and 2003, based on the 1999 curriculum, have not been revised either. Similarly, in Cyprus the current textbook has been used since 1998.

Given these difficulties, we decided to analyze the treatment of addition and subtraction of fractions in fourth grade in Cyprus and Taiwan because substantial, formal work with algorithms was done on the topic in this grade in both countries. In Ireland the topic is not formally introduced until fifth grade, so we looked at how the topic was treated in fifth grade there. By selecting these grade levels we could observe how the topic was sequenced within each textbook, how worked examples were developed, the fractional constructs used, the level of cognitive demand inherent in the exercises assigned to students, and the type of answers required from students. Such insights would allow comparisons of how the topic was treated across countries and could offer benchmarks for comparisons with other countries.

Our second sampling decision concerned the selection of textbooks for analysis. This was not a concern for Cyprus, where only one textbook was available. In contrast, in Ireland and Taiwan multiple textbooks were available to schools and teachers. The two Irish textbooks reviewed in this study were the most widely used across the country; similarly, the two Taiwanese textbooks reviewed here were among the three most widely used in the Taiwanese schools. Because we chose typical textbooks for our analysis, our findings should be considered *illustrative* of how one mathematical topic is presented by textbooks in each country (Patton, 2002).

Having selected the topic and the grades to study, we identified the pages in each textbook where addition and subtraction of fractions were considered. We photocopied the pages, and translated the Cypriot and Taiwanese sections into English so that all authors could analyze the content of all textbooks and so that inter-rater reliability could be gauged. We listed the topics introduced in each textbook and their order of appearance and compiled the topics into a table. We examined which topics were present in each textbook and how they were sequenced.

For the remaining four criteria (worked examples, fraction constructs, potential cognitive demands, and type of answers) we followed a combination of top-down and bottom-up approaches for establishing the meaning and the coding for our categories. We relied mostly on the literature to derive meaning for our categories (a top-down approach), but we also considered the actual content in the textbooks (bottom-up approach); we went back and forth between the literature and the textbooks in order to create a coding system that was consistent with the literature and with the content of the textbooks. Three principles guided this iterative process: (1) building on the existing literature of textbook analysis and the topic under consideration, (2) ensuring that the framework was sensitive to key characteristics of all textbooks being analyzed, and (3) respecting the mathematical integrity of the criteria for analysis.

Drawing from Watson and Mason (2005), we considered worked examples as those portions of the textbook "that demonstrate the use of specific techniques" (p. 3); in our case the addition and subtraction of fractions. We did not limit our exploration to examples that included solution steps and the final solution (cf. Renkl, 2002, p. 529) because, as explained above, an initial exploration of the textbooks of the three countries pointed to differences in the level of completeness of worked examples. This approach resonates with how worked examples were considered in other studies (e.g., Valverde et al., 2002).

In analyzing the worked examples, we followed a grounded-theory approach (Strauss & Corbin, 1998). First, we considered the procedures for which a worked example was included in the textbooks and how these worked examples were sequenced in the textbooks. Comparing the worked examples and identifying how the examples attempted to facilitate student learning led us to generate a set of questions that guided subsequent analysis. Thus, in addition to considering the constructs and the representations in the worked examples, we asked: Do these examples state a rule or the steps for a procedure to follow? Do they present more than one method for the same procedure (e.g., for adding two mixed numbers), and if so, do they build connections between the different methods? What types of other graphical displays (i.e., graphics other than representations of mathematical ideas, such as cartoons, photographs, or other non-mathematical drawings) do these examples employ? And, are these examples situated in a mathematical—more abstract—context or are they embedded in situations closer to students' daily experiences (e.g., sharing food)?

Exercises/problems were defined as the textbook segments that require students to answer one or more questions, apply a procedure, or solve one or more problems. We initially defined a task according to the highest level of numbering of the exercises/problems as designated by the textbook authors in each country. Because this resulted in notable differences in the length and requirements of tasks in the textbooks of the three countries,[6] we refined our definition of "tasks" as the questions asked in the exercises/problems. In most cases, these corresponded to the second level of numbering, but in a few cases we found questions that were not numbered, but still counted as tasks.

[6]Defined by the highest level of numbering, the tasks in both Taiwanese textbooks tended to include only one or two exercises/problems (more than 90% of the textbook tasks). In contrast, more than 40% of the tasks in the CT and more than 50% of the Irish textbooks included tasks with five or more exercises/problems.

In analyzing the fraction constructs, we used definitions from previous studies (e.g., Lamon, 1999). In particular, the part-whole construct referred to a continuous quantity or a set of discrete objects partitioned into parts of equal size (e.g., a pizza divided into *n* equal parts); the ratio construct was taken as a multiplicative comparison between two quantities (e.g., a comparison between the boys and the girls of a class); the operator construct reflected a function that transforms line segments, figures, or numbers (e.g., how many times should 9 be increased to get 15?); the quotient construct corresponded to considering a fraction the result of a division of two whole numbers (e.g., how much pizza would four friends get if they shared 5 pizzas evenly?); and the measure construct related to identifying fractions as numbers or associating fractions with the measure assigned to some interval (e.g., locating fractions on number lines).[7] After an initial inspection of the textbooks, we included an additional code that combined two of the above constructs. We named this code "part-whole and operator." In making decisions about what fraction construct was at stake in a given task, we relied on the accompanying representations or the problem context. When the tasks lacked either one or both of these, we assigned no construct.

We analyzed the potential cognitive demands of each task using the *Task Analysis Guide* (Stein, Smith, Henningsen, & Silver, 2000, p. 16). According to this guide, *memorization* tasks ask students to reproduce previously learned facts, rules, formulas, or definitions and can be solved without using procedures. Tasks requiring *procedures without connections* to meaning are usually algorithmic and have no connection to the concepts or meaning that underlie the procedures being used. Tasks requiring *procedures with connections* to meaning focus students' attention on the meanings and concepts underlying the procedures needed to solve the task. In solving these tasks, students cannot mindlessly use a learned procedure. According to the classification proposed by Stein et al. (2000), these tasks require more than the application of a well-established procedure. Tasks characterized *as doing mathematics* require complex and non-algorithmic thinking because of the unpredictable or not easily discernible nature of their solution process. The first two categories correspond to tasks of lower level demands, and the other two to tasks of higher cognitive complexity.[8]

Finally, we classified each task according to the type of response required from students. Following previous studies (e.g., Mayer et al., 1995; Schmidt et al., 1997), we examined whether a task explicitly requires students to (1) provide only an answer (numerical answers or numerical expressions), (2) explain their answer or the process they followed to get that answer, and (3) justify the reasonableness of the approach they pursued in solving the task or the rationality of their answer. After the first round of analysis, we added a fourth category to account for tasks that asked students to provide both an answer and the mathematical sentence used to get this answer. A list of the criteria we used to code the textbooks of the three

[7]A more detailed description of these constructs as used in this study appears in Charalambous and Pitta-Pantazi (2007).

[8]We are aware that the cognitive demands of the tasks depend on students' prior experiences. However, we decided not to include such considerations in our analysis because exploring the textbooks of previous grades was not considered sufficient for determining students' prior knowledge; the presence alone of a topic in the textbook of previous grades does not ensure students' prior knowledge of that topic. Because we decided not to take into consideration students' prior experiences and because we do not know how these tasks might actually be enacted during a lesson, we talk about *potential* cognitive demands.

TABLE 2
Criteria Used in the Coding System

Topics
Adding proper fractions
Adding a whole number to a proper fraction
Adding a proper fraction to a mixed number or improper fraction
Adding two mixed numbers or improper fractions
Subtracting a proper fraction from a whole number
Subtracting a proper fraction from a proper fraction
Subtracting two mixed numbers
Subtracting a proper fraction from a mixed number or improper fraction
Sequencing
Sequencing of the topics
Topics preceding the addition and subtraction of fractions and mixed numbers
Worked Examples
Frequency
Procedures outlined
Methods per procedure
Building connections
Constructs
Type of representation
Context
Constructs
Part-whole
Ratio
Operator
Quotient
Measure
Combinations of the above
Potential Cognitive Demand
Memorization
Procedures without connections to meaning
Procedures with connections to meaning
Doing mathematics
Type of Response
Only an answer
Answer and mathematical sentence
Explanation
Justification

countries appears in Table 2 and examples of the coding as applied to tasks are provided in the Appendix.

Reliability of the Coding System

We took several steps to ensure the reliability of the coding process. First, we selected particular examples from each country to determine if the categorizations proposed in the framework were capturing differences among the three countries; we chose tasks we thought difficult to categorize.

We selected 10% of the sample of pages from each textbook and coded the tasks along the cognitive demands, type of response, and fraction constructs. In the first trial, we found outstanding reliability for cognitive demands for Ireland and Taiwan (Cohen's κ[9] = 1.0 in both cases) and type of response for Cyprus and Ireland (κ = 1.0 in both cases); we had substantial reliability for fraction constructs for Taiwan (κ = 0.73). For the rest—cognitive demand in Cyprus, type of response in Taiwan, and construct in Ireland and Cyprus—we had low reliability.

The discrepancies in cognitive demands occurred in distinguishing procedures with connections from procedures without connections. Stein et al. (2000) suggest that tasks involving a well-established procedure be classified as procedures without connections. But when conducting a textual analysis, it is difficult to determine how established a procedure is for a particular group of students. Hence, we decided to relax this condition and to code tasks as procedures without connections only if they had an algorithmic character and made no connections to the underlying meaning of the procedures of adding and subtracting fractions. The low reliability for the type of response for the Taiwan sample occurred in tasks that required students to give not only the answer but also the sentence that modeled the problem; it was an omission of one of the coders. The discrepancies in the fractional constructs were due to the coders' slightly different interpretations of the operator and the part-whole constructs. We thus refined our description of these constructs to eliminate ambiguities.

To test these adaptations we selected another 5% of pages, choosing again cases we thought would be most difficult to code. In this second test, we had outstanding reliability for the type of responses and the fractional construct ($\kappa \geq 0.90$), but low to moderate reliability for cognitive demands ($0.36 \leq \kappa \leq 0.70$).

The discrepancies in the cognitive demands again emerged in distinguishing between procedures without connections and procedures with connections. This time, most of the disagreements concerned the extent to which tasks drew connections to the meaning of the underlying procedures or not. Hence, we agreed that, when in doubt,[10] we would err toward assuming that the tasks required making connections to meaning. This helped bring about consensus. In light of these results, we coded all tasks in the textbooks and met to discuss tasks for which raters had difficulties coding. We resolved these through consensus.

FINDINGS

This section reports our findings with respect to the six criteria that informed our analysis: topics and their sequencing, (Horizontal dimension), worked examples, fraction constructs, potential cognitive demands, and type of responses (Vertical dimension).

[9]We used Cohen's κ because it allows assessing inter-rater reliability when there are two coders and the variables have several categories. Because this coefficient does not count agreement that is simply due to chance, it is more stringent than simply calculating the rate of agreements to the total of agreements and disagreements. According to Landis and Koch (1977), κ = 0.40 to 0.59 reflects moderate inter-rater reliability, 0.60 to 0.79 substantial reliability, and 0.80 or greater outstanding reliability.

[10]For instance, some tasks in TTA (vol. 2, p. 72) presented students with strips and asked them to figure out the combined length of two strips joined end-to-end or the difference in length between two strips. Although these tasks can be solved by a simple addition or subtraction, we decided to code them as procedures with connections because they implied some connection to meaning.

Covered Topics and their Sequencing

All the textbooks introduced similar content prior to the sections on the addition and subtraction of fractions (concept of fractions, equivalent fractions, fraction simplification, converting improper fractions to mixed numbers or vice versa, and comparing and ordering fractions). Moreover, in all five textbooks addition and subtraction topics were interspersed; that is, none of the textbooks first introduced all possible operations on the addition of fractions and then moved to the subtraction of fractions. In other words, the content was organized by the nature of the numbers considered and not by type of operation (see Figure 2).

We found two differences among the three countries. First, in the textbook from Cyprus and one of the textbooks from Taiwan (TTA) addition and subtraction of fractions and mixed numbers with like denominators was included in the fourth-grade textbooks, whereas the additive operations on fractions and mixed numbers with unlike denominators were postponed for the next grade. Both Irish (fifth-grade) textbooks presented addition and subtraction of fractions and mixed numbers with similar and different denominators. Only TTB included the addition and subtraction of fractions with dissimilar denominators in fourth grade.

Second, the Irish textbooks followed a slightly different sequence for the content compared to the other textbooks. Both Irish textbooks began with addition and subtraction of fractions with similar and dissimilar denominators. Then, both of them introduced additive operations on mixed numbers whose fractional part has different denominators; different denominators were the norm in these mixed-number tasks with only a few involving operations on fractions with the same denominator. In contrast, the CT and TTB both began with addition and subtraction of fractions with similar denominators and then moved to the addition and subtraction of mixed numbers whose fractional part has the same denominator. These two textbooks covered all possible cases of the addition and subtraction of fractions with the same denominator: addition and subtraction of proper fractions; addition (subtraction) of a proper fraction to (from) a whole number, a mixed number or an improper fraction; and addition and subtraction of two mixed numbers or improper fractions. After covering these cases, the TTB shifted to the addition and subtraction of fractions with dissimilar denominators. Like TTB, TTA first presented the addition of proper fractions, but then considered the subtraction of a proper fraction from a mixed number or an improper fraction, before considering the subtraction of a proper fraction from another proper fraction or from a whole number; the addition and subtraction of mixed numbers was introduced last.

Worked Examples

Altogether the textbooks analyzed had 49 worked examples. Although we found recurring patterns among the worked examples, overall more differences than similarities were found. The similarities related to the procedures demonstrated, constructs, and representations used.

Procedures

In all textbooks the most common procedures demonstrated in the worked examples were adding two proper fractions, adding two mixed numbers (or improper fractions), subtracting two proper fractions, and subtracting two mixed numbers (or improper fractions). We also found

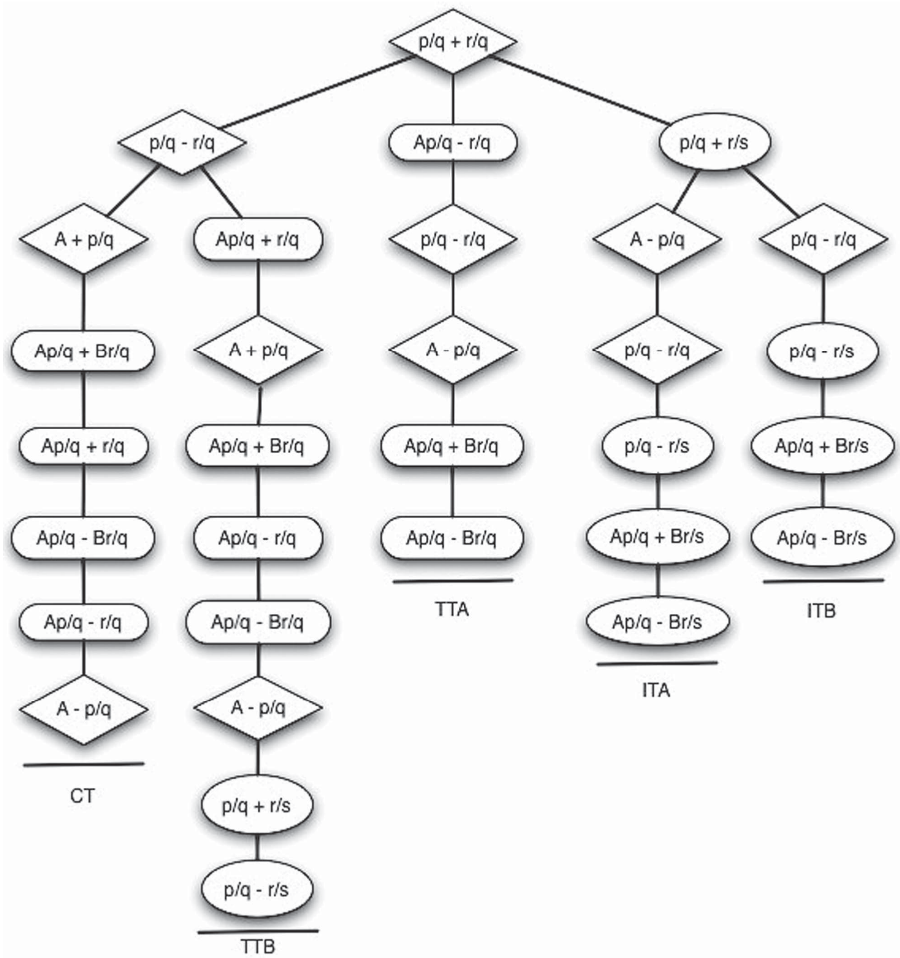
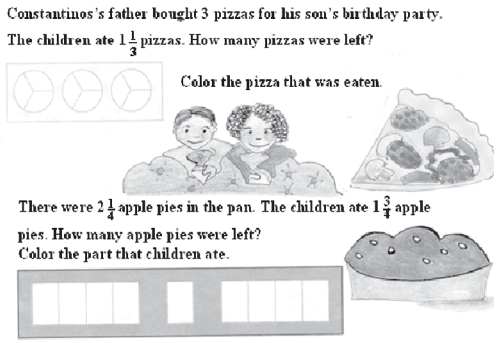


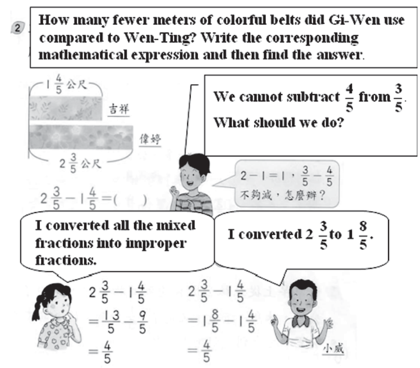
FIGURE 2 Topics and their sequence in the textbooks of the three countries.

Note: p, q, r, s, A and B are positive natural numbers, $p < q$, $r < q$, and $r < s$; Ap/q and Br/q represent mixed or improper fractions. The diagram shows the different order in which operations with proper and improper fractions or mixed numbers with similar and dissimilar denominators were sequenced in each textbook analyzed. Notice that only the Irish textbooks and TTB deal with addition and subtraction of fractions with unlike denominators, and that the textbook from Cyprus and both Taiwanese textbooks deal with several cases of addition and subtraction of fractions involving like denominators.

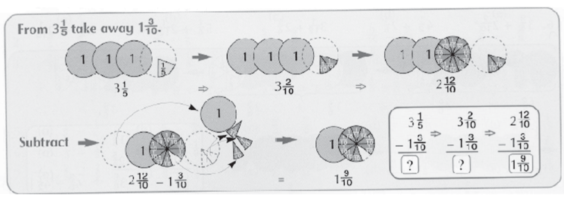
examples of adding a whole number to a proper fraction (2 each in CT and TTB) and examples of subtracting a proper fraction from a whole number (1 each in CT, ITA, TTA, and TTB). We found no examples where proper fractions were added to or subtracted from improper fractions. The Irish textbooks and one Taiwanese textbook, TTB, had worked examples where the denominators



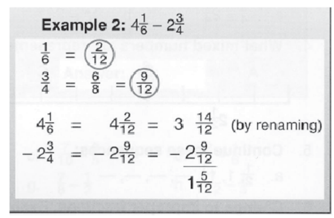
(a) Cypriot Textbook (CT, vol. 4, p. 47)



(b) Taiwanese textbook (TTA, vol.2, p. 85)



(c) Irish Textbook (ITA, p. 50)



(d) Irish Textbook (ITB, p. 50)

FIGURE 3 Worked examples on the subtraction of mixed numbers. Reproduced with permission from (a) the Ministry of Education and Culture, Cyprus; (b) Kang Hsuan Educational Publishing Group, Taiwan; (c) C. J. Fallon Ltd, Ireland; and (d) Folens Publishers, Ireland.

differed. All textbooks included examples of subtracting two mixed numbers where the fractional part of the subtrahend was greater than that of the minuend (see Figure 3).

Constructs

The part-whole construct was present in most worked examples in the textbooks analyzed (see Figure 3a and Figure 3c). In ITB, however, no construct was specified in 8 of the 10 worked examples (Figure 3d), but when a construct was used it was part-whole. Although part-whole was the most common construct in both Taiwanese texts, in 2 of 5 examples in TTA and in 8 of 13 examples in TTB, the part-whole construct was accompanied by the operator construct. Consider the following example:

> One box has 15 jellies. There was one box of jellies and Xai-Guan ate 3/5 of it. How much of a box of jellies is left now? How many jellies does this fraction correspond to? (TTA, vol. 2, p. 83)

The textbook example suggested two ways of approaching this problem, the first using the part-whole interpretation of fractions, namely: "One box of jellies equals 5/5 boxes of jellies. 5 of 1/5 minus 3 of 1/5 is equal to . . .", and the second using the operator construct, that is: "1/5 boxes is 3 jellies. 3/5 boxes are 3×3 = 9 jellies. The rest is . . .". TTA also included the measure construct of fractions in two worked examples—the only textbook of the five to do so (see Figure 3b).

Representations

The predominant representation in worked examples across all textbooks was an area model—although slight variations to this pattern were found. Area models can be seen in Figure 3a (from CT), which features pizzas and apple pies, and Figure 3c (from ITA), featuring a more abstract series of representations. In CT, every worked example was accompanied by an area model (two-thirds of them circular and one-third rectangular). In ITA, circular representations (like those in Figure 3c) accompanied all 12 worked examples, whereas in ITB only 2 of the 10 worked examples had area models (the remaining 8 examples had no pictorial representation, see Figure 3d). TTA had five worked examples; two had no representation, two had linear representations, and one had a circular area representation. In TTB, 2 of the 13 worked examples used circular area models; three used linear representations, and four used discrete sets. The remaining four worked examples were accompanied by no representations. Overall, the two Taiwanese textbooks used linear representations and discrete-set representations; the other textbooks favored area models whereas linear and discrete set representations were absent.

In sum, with only a few exceptions, most textbooks showcased similar procedures in their worked examples; they emphasized the part-whole construct and favored area models as a representation. In contrast, we found major differences regarding the completeness of worked examples, the contexts in which worked examples were placed, the frequency and type of graphical displays (other than representations), and the number of problem solution methods. We detail these differences below.

Completeness

In both Irish textbooks every worked example included the solution (as in Figure 3c and Figure 3d), whereas examples that included the solution were uncommon in the textbooks from both Cyprus (2 out of 7) and Taiwan (1 out of 5 in TTA and 6 out of 13 in TTB). Figure 3a—from CT—shows one worked example that did not include the solution; in this example, students are required to color pizzas and apple pies in a diagram to figure out the solution to the given problems. All Irish worked examples explicitly illustrated the steps to be followed when completing procedures. In contrast, although two-thirds of the worked examples in TTB explicitly illustrated the steps, in TTA and CT only about half the worked examples illustrated the steps of the procedure being explained. Thus, in Figure 3a, for instance, students must decide which pizzas to color in order to shade 1 1/3; also, the example in Figure 3b does not remind students *how* to convert mixed numbers into improper fractions but simply states it as something that was done. In contrast, the example in Figure 3c, presents the student and teacher with a series of representations explaining each step of the procedure.

Contexts

The Irish textbooks differed from those in the other two countries in terms of the context around which the worked examples were built. Most worked examples in the Irish texts were set in exclusively mathematical contexts (11 out of 12 in ITA and 8 out of 10 in ITB, see Figure 3c and Figure 3d). In the other two countries, worked examples were more often embedded in “real-world” contexts (6 out of 9 in CT; 5 out of 5 in TTA and 13 out of 13 in TTB).

Graphical Displays

No graphical displays were found in the worked examples analyzed in the Irish textbooks and only two were found in the Cypriot text (e.g., Figure 3a). But several graphical displays accompanied every worked example in both Taiwanese textbooks. Most graphics were cartoon pictures of students with speech bubbles emanating from their mouths—examples of what Davis (2007) called representations of spoken-language—explaining the steps of a procedure (Figure 3b). Other graphical displays in TTB contained reproductions of text on a chalkboard, possibly used to model what a teacher might write in class or what students might write in their notebooks. Other frequently used graphical displays in the Taiwanese worked examples were used to enhance the "real-world" context of the examples.

Solution Methods

The final difference was in the number of methods demonstrated in worked examples. In all worked examples from Cyprus and Ireland only one method was shown. In contrast, only 2 of the 18 Taiwanese worked examples (both in TTB) demonstrated a single method. Taiwanese worked examples provided at least two methods (two TTB examples gave three methods). For example, in Figure 3b, a student faces the problem of subtracting 4/5—the fractional part of a mixed number—from 3/5, the fractional part of another mixed number. Two alternative methods are given: change both mixed numbers to improper fractions; convert (trade) one unit from the mixed number minuend to a fraction and add it to the fractional part of the minuend.

In sum, worked examples in Irish textbooks differed from the Cypriot and Taiwanese textbooks regarding the completeness of solutions and the use of contexts, with the Irish textbooks offering complete solutions within exclusively mathematical contexts. Worked examples in Taiwanese textbooks contained more graphical displays and offered more solutions than the Cypriot and Irish textbooks. These differences might translate into different opportunities to learn for students and teachers as they engage with the content of these textbook sections.

Fraction Constructs

Now we shift to the textbook tasks and in particular to the kind of constructs elicited in those tasks. No construct was assigned to most tasks in the Irish textbooks (97% for ITA and 80% for ITB) and to about half of the tasks in CT (47%), because the tasks were not accompanied by any representation or set in any problem context. In contrast, the fraction construct could be determined for most tasks in the Taiwanese textbooks (about 65% for TTA and 80% for TTB, see Table 3).

As might be expected, the part-whole construct was identified in tasks in all three countries, with this being the dominant construct among tasks where constructs were identified in the Cypriot and Irish textbooks; indeed it was the only construct present in ITA. In contrast, the part-whole construct appeared less frequently in Taiwanese textbooks tasks (6% and 14%, for TTA and TTB, respectively).

Only a handful of tasks in the Cypriot and Irish textbooks and TTB used the measure construct (about 5% for CT, 9% for ITB, 6% of TTB), whereas 31% of tasks in TTA used this construct (see Table 3). These tasks typically provided students with the measure of a line segment, the weight of different goods, or the volume of liquid in volumetric glasses.

TABLE 3
Frequencies and Percentages of Tasks Using Each Construct

	Cypriot Textbook		*Irish Textbooks*				*Taiwanese Textbooks*			
	CT N = 211		*ITA N = 233*		*ITB N = 81*		*TTA N = 62*		*TTB N = 106*	
Construct	*#*	*%*	*#*	*%*	*#*	*%*	*#*	*%*	*#*	*%*
Part-whole	93	44	7	3	7	9	4	6	15	14
Ratio	0	0	0	0	0	0	0	0	0	0
Operator	5	2	0	0	0	0	2	3	10	9
Quotient	0	0	0	0	0	0	0	0	0	0
Measure	10	5	0	0	7	9	19	31	6	6
Part-whole and operator	4	2	0	0	0	0	16	26	45	43
Part-whole and quotient	0	0	0	0	0	0	0	0	10	9
Part-whole and measure	0	0	0	0	2	2	0	0	0	0
Not classified	99	47	226	97	65	80	21	34	20	19

The Taiwanese textbooks featured tasks in which both the part-whole and operator constructs could be elicited in the same task (26% of tasks in TTA and 43% in TTB). Such tasks provided information on the number of discrete objects in each unit, so either construct could be used in solving the problem:

> One bag has 30 peaches. Feng-Hau has 3/10 bags of peaches and Shai-Yuh has 9/10 bags. How many fewer bags of peaches has Feng-Hau compared to Shai-Yuh? (TTB, vol. 1, p. 13)

Students can solve the problem by using the part-whole construct (i.e., Feng-Hau has 9/10–3/10 = 6/10 bags fewer than Shai-Yuh); but including the number of peaches in each bag gives students the opportunity to solve the problem following a more complex, yet legitimate, approach, that draws on the operator construct:

> Feng-Hau has 3/10 bags of 30 peaches, which corresponds to 9 peaches; Shai-Yuh has 9/10 bags of 30 peaches which corresponds to 27 peaches; thus, Feng-Hau has 18 peaches less than Shai-Yuh, which is the same as 18/30 bags or 6/10 bags of peaches.

Only 2% of the CT's tasks combined a part-whole and an operator construct; such tasks were absent from both Irish textbooks. In addition, one Taiwanese textbook (TTB) used the quotient construct in a task related to the addition and subtraction of fractions:

> One box has 6 popsicles. Divide one box of popsicles into equal parts. How much of a box of popsicles is one popsicle equal to? How many boxes are there if we add 2 boxes and 5 of 1/6 boxes of popsicles? How many boxes are there if we add 3 boxes and 4 of 1/6 boxes of popsicles? (TTB, vol. 1, p. 49)

No textbook included addition or subtraction tasks that used the ratio construct of fractions.

Tasks in the Taiwanese textbooks exhibited a unique feature regarding how questions were asked. The tasks included what we dubbed *twin* questions: two questions that always appeared in sequence and asked students to consider a relative amount (i.e., "how much of") and an absolute amount (i.e., "how many"), as in the following task:

> One bag has 12 tomatoes. 2/6 bags were used to prepare a soup. How much of a bag of tomatoes was left? How many tomatoes is this equal to? (TTA, vol. 2, p. 83)[11]

The first question prompts students to consider a relative amount—namely, how much of a bag of tomatoes was left—after using a portion of it (1 – (2/6) = 4/6). The second question, which often requires the employment of the operator construct, prompts students to calculate an absolute amount, in this case the number of tomatoes left (4/6 of 12 tomatoes). Putting the questions together gives students the opportunity to link the relative to the absolute amounts, which is considered important for students' understanding of fractions (Lamon, 1999).

Another characteristic observed in both Taiwanese textbooks was their use of the concept of the *unit fraction*.[12] For instance, after introducing the example below, TTB asks students to start with 1/5 and add another 1/5 each time, till they get to the whole unit:

> One box has 10 bags of cookies. Five children share this box equally. How much of a box of cookies does each child get? How many bags of cookies does each child get? Explain how you get the answers (TTB, vol. 1, p.15).

Other tasks also prompt students to consider a part of the whole as consisting of a number of fraction units as in the task in Figure 4. In this task, the white cube represents the unit fraction. The question "How much of the blue block is a white cube equal to?" may lead to establishing that a white cube is equal to one-ninth of the block of blue cubes. Then, the two questions "How much of the blue block are four white cubes equal to?" and "How much of the blue block are 4 of 1/9 of the blue block equal to?" prompt students to connect the four-ninths of the block of blue cubes to four white cubes, or alternatively to four 1/9-units (i.e., $4/9 = 4 \times 1/9$). The last set of twin questions ("If we have 4 of 1/9 of the blue block and add another 1/9 of the blue block, how many of 1/9s of the blue block will we get? How much of the blue block will this be equal to?") prompts students to consider the sum of adding blue cubes to the existing block of cubes both as a part of the whole (e.g., 5/9 of a block of blue cubes) and as a set of unit fractions (e.g., a set of 5 1/9s).

Across all textbooks we found that most tasks (in which a construct is implied) used a part-whole construct. However, the Taiwanese textbooks tended to systematically incorporate questions that highlight constructs besides the part-whole construct, such as the measure and the operator constructs; they also introduced the notion of unit fraction.

Potential Cognitive Demands

Tasks in both Irish textbooks appeared to be organized on a continuum, from tasks with lower demands to tasks with higher demands, whereas in the Cypriot and Taiwanese textbooks tasks of lower and higher demands were interspersed throughout the relevant sections. We classified more than 85% of tasks in the Cypriot and the Irish textbooks as representing low cognitive

[11]In Chinese, these twin questions are both phrased using the terms "how many?" (i.e., "how many bags of tomatoes?" and "how many tomatoes?"). The original version of this problem as it appears in Chinese is "一袋蕃茄有12個，用了2/6袋煮湯後，再用多少袋蕃茄，就會把一袋蕃茄都用完呢？是多少個？"

[12]The concept of unit fraction, as used by ancient Egyptians, integrates three fundamental ideas—equipartitioning, unit, and quantity—all of which underlie all fraction constructs (cf. Carpenter, Fennema, & Romberg, 1993, p. 3). Brousseau (1997, pp. 104–105) also comments on the importance of unit fractions for the addition of fractions.

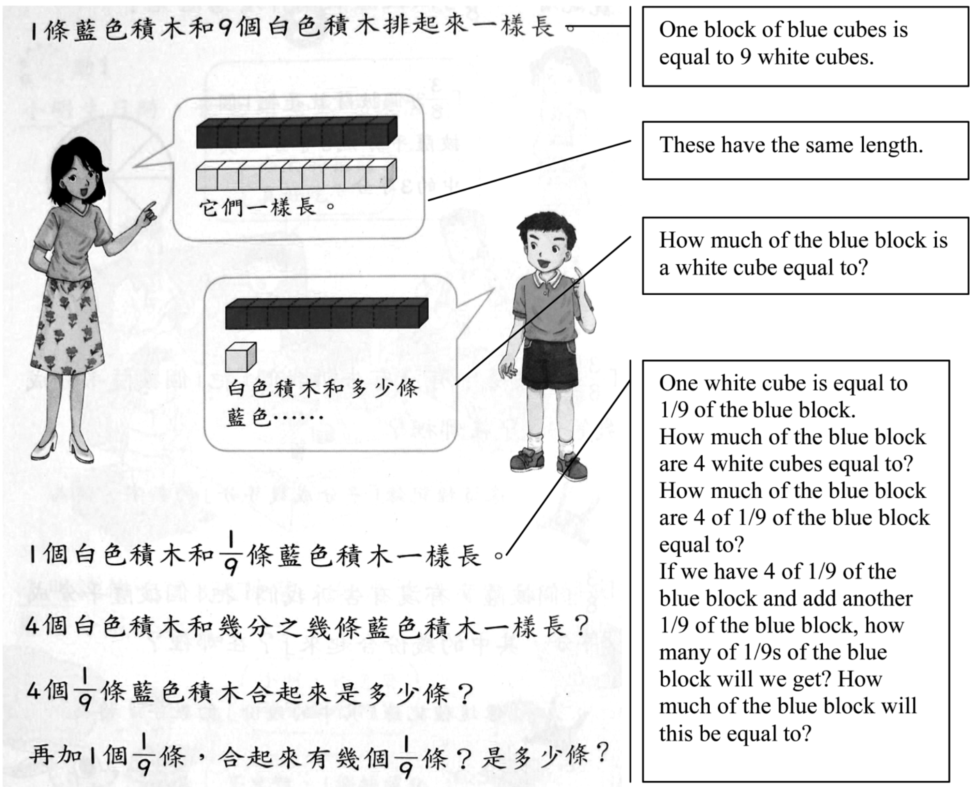


FIGURE 4 The unit fraction in the Taiwanese Textbooks (TTB, vol. 1, p. 12). Reproduced with permission from Han Lin Publishing Co. Ltd, Taiwan.

demand (i.e., procedures without connections), whereas we classified about 71% of the tasks in the TTA and 81% in the TTB as having high cognitive demands (procedures with connections or doing mathematics, see Figure 5). Moreover, *procedures without connections* tasks in the Taiwanese textbooks were generally of a different nature than those included in the Cypriot and Irish textbooks. For example, besides asking students to find the sum or the difference of two fractions or mixed numbers (e.g. 6/7 + 4/6 = ?), 11% of the tasks in the TTB and less than 1% of the tasks in the CT required students to find the missing addend or minuend/subtrahend (e.g., 8 5/12 + ? = 10 1/12; TTB, vol. 2, p. 23). Studies by Fuson (1992) and Marshall (1995) showed that missing addend or minuend/subtrahend tasks impose higher demands on students than tasks in which the unknown quantity is the sum or difference.

Type of Response

Textbooks differed in the responses they required from students. Whereas all tasks in the Cypriot and the Irish textbooks required single answers (Figure 6), about 29% of tasks in

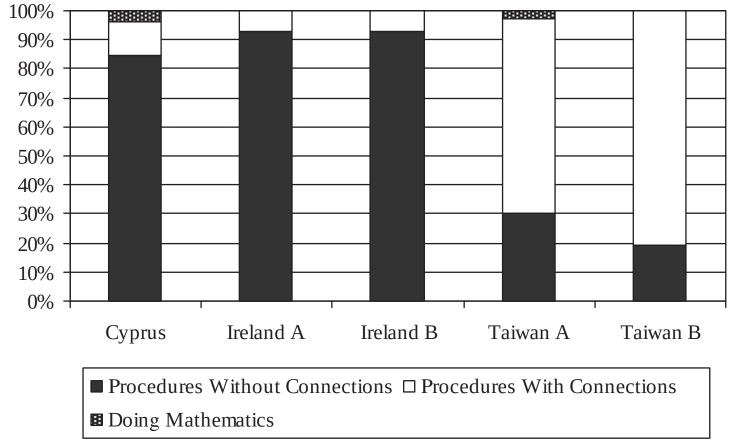


FIGURE 5 Percentages of tasks exhibiting each cognitive demand.
Note. No memorization subtasks were identified.

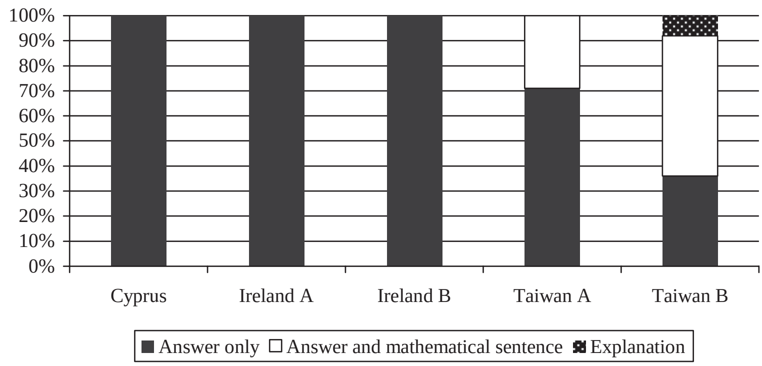


FIGURE 6 Percentages of tasks exhibiting each type of response.
Note. No subtasks asking students to justify their answer were identified.

TTA and 56% of tasks in TTB required students to represent the underlying structure of the problem in a mathematical sentence. Similarly, in about 8% of tasks in TTB students were asked to explain the approach used to solve the task. Consider, for example, the following problem:

> The length of the red rope is 2 37/100 meters. The length of the white rope is 14/100 meters. How long are the two ropes altogether? Write the mathematical sentence for this problem and give the final result in the form of a mixed number. Explain your thinking process in solving this problem (TTB, vol. 1, p. 76).

No such requirement was observed in the other textbooks.

DISCUSSION

In this study we used a new framework to compare and contrast general and particular aspects of textbooks from three countries and identify how the textbooks could afford opportunities for supporting student learning. We organize the discussion of our findings by returning to our research questions:

1. What similarities and differences can be observed in the presentation of addition and subtraction of fractions in primary mathematics textbooks in Cyprus, Ireland, and Taiwan?
2. What expectations of student performance on addition and subtraction of fractions are embedded in primary textbook tasks in Cyprus, Ireland, and Taiwan?

The Presentation of the Addition and Subtraction of Fractions in the Textbooks

In order to describe the presentation of addition and subtraction of fractions, we discuss the findings that correspond to two criteria from the horizontal dimension (topics and sequencing) and two from the vertical dimension (constructs and worked examples). By looking closely at what is presented we can begin to identify what students would "learn if their mathematics classes were to cover all the textbook sections in the order given" (Mesa, 2004, p. 256).

The analyzed textbooks present roughly the same content in somewhat similar order. In two countries—Cyprus and Taiwan—the content is organized by the nature of the numbers (i.e., similar or dissimilar denominators, mixed numbers) rather than by the operations (i.e., addition and subtraction). This organization appears to assume students' proficiency with operations with whole numbers from earlier grades and focuses on how the nature of numbers might increase the complexity for students. Drawing students' attention to specific aspects of the fractions (e.g., their denominators) and compartmentalizing the procedures associated with them might make the content more manageable, both for teachers and students. Teachers might use processes associated with fractions with similar denominators to build students' confidence before moving to fractions with dissimilar denominators; as students become more competent and confident with procedures including similar denominators, these procedures might be reified into an object—to use Sfard's (1991) terminology—on which students can draw when dealing with the new procedures, namely those related to working with dissimilar denominators. Unlike Cyprus and Taiwan, the Irish textbooks presented fraction operation tasks with similar and dissimilar denominators together. By doing this, the Irish authors organize the content by the operation rather than by the nature of the numbers. Ordering tasks by operations (addition first, subtraction next, with all numbers) suggests to us that the textbooks are responding to the mathematical, rather than the cognitive, perspective of learning fractions. This mathematical perspective maintains that the rational numbers are in a field (a field is closed under addition and subtraction). In contrast, the cognitive perspective acknowledges that some aspects of the representations of numbers (mixed, similar, or dissimilar denominators) might present cognitive difficulties, even though all numbers belong to the same set (i.e., they are rational). These decisions are not trivial; sequencing content must have some underlying principled decision-making process that takes into account mathematical and pedagogical considerations, to psychologize the subject-matter in ways that render it teachable and comprehensible by the child (Dewey, 1906, pp. 30, 38). Such a transformation of the content will affect what happens in the classroom, even though we cannot assume that teachers will follow the textbook as it is sequenced.

Nevertheless, how the content is organized might have less impact on the quality of student learning than other accompanying criteria such as worked examples and the cognitive demands of tasks. What content is offered will matter little if the content development in the textbook does not explicitly address core aspects of the notion. Likewise, the order is less important when the tasks themselves require students to do little more than memorize or repeat procedures.

Previous studies that focused on the teaching and learning of fractions (Baturo, 2004; Boulet, 1998; Lamon, 1999) have emphasized that engaging students with multiple fraction constructs catalyses learning, not only because each fraction construct captures part of the broader notion of fractions but also because moving from one construct to another reinforces understanding. Furthermore, the dominant part-whole interpretation of fractions has limitations (e.g., the whole cannot exceed the number of partitions) which, according to some researchers (e.g., Smith, 2002), might impede students' understanding of and work with improper fractions. Our analysis pointed to key differences in how textbooks in the three countries craft opportunities for their students to work with multiple fraction constructs when adding and subtracting fractions. Whereas the part-whole was the main construct used to develop addition and subtraction of fractions in the Cypriot and both Irish textbooks, the operator construct accompanied this construct in the Taiwanese textbooks. With the exception of the ratio construct, which was not present in any of the textbooks studied, the other constructs (i.e., operator, measure, and quotient) were also more common in the Taiwanese textbooks than in the other textbooks. Interestingly, most tasks in TTA emphasized the measure construct, a construct that is pivotal for understanding both the notion of fractions and their addition and subtraction (Lamon, 2001; Keijzer & Terwel, 2001). This difference is not trivial, given that Behr and associates' (1983, p. 100) model presents the measure construct alongside the part-whole construct as the most "natural" path to understanding additive operations on fractions.

Tasks in the Taiwanese textbooks drew largely on unit fractions, a notion that has been considered a link among different constructs (e.g., part-whole, the measure, and the quotient) and a catalyst for students' transition from whole to rational numbers (Carpenter, Fennema, & Romberg, 1993; Mack, 2001). Equally important for students' transition to rational numbers is the notion of relative quantity (Lamon, 1999). Unlike the Cypriot and Irish textbooks, the Taiwanese textbooks systematically included tasks that considered both the absolute and the relative amounts by prompting students to respond to what we identified as twin questions: the "how much" and the "how many" aspects of a given situation.

Although all textbooks offered worked examples, notable differences existed across the countries. Worked examples in the Taiwanese textbooks used more diverse fraction constructs and representations than the other textbooks. Taiwanese textbooks tended to offer worked examples using the operator and the measure constructs accompanied by discrete-set or continuous linear representations; they also modeled more than one method for adding and subtracting fractions by using speech bubbles. It appears that these strategies serve the dual purposes of acquainting students with each interpretation of fractions and prompting students to build connections among them. In general, the Irish textbooks provided a full and clear sequence of steps for students to follow when adding and subtracting fractions, while examples in the other textbooks were partially complete, presumably expecting students to finish the process.

These findings suggest that choices made about topics and sequencing, but most importantly, the diversity of constructs and worked examples impose different demands on students who use the textbooks in these countries. As such, these findings seem to support the argument made earlier in the paper about the need to pursue both a horizontal and a vertical analysis of textbooks, as each type of analysis helps identify different textbook features that are conceivably important for student learning.

The selection and sequencing of topics not only frame what is to be learned but also could facilitate or impede this learning, especially if one considers prior studies suggesting that the relative ordering of topics can nurture student difficulties and misconceptions (e.g., Resnick et al., 1989). For example, when adding fractions, students tend to add both the numerators and the denominators. It is therefore a matter for further research whether an instructional sequence that first exposes students to adding similar-denominator fractions and then shifts to adding dissimilar-denominator fractions (as in the Cypriot and the Taiwanese textbooks) is better suited to counteract that tendency rather than an instructional sequence that exposes students to adding similar- and dissimilar-denominator fractions simultaneously (as in the Irish textbooks). But the selection and ordering of topics can tell only part of the story of the learning opportunities that textbooks can craft for students.

When the content and the examples use multiple constructs with many representations and more than one solution method, students have more opportunities to develop more complex, connected, and robust understandings of fractions. Also, textbooks that are less explicit about the solution process in worked examples impose more demands on students as they are required to extrapolate from the information given to finish a procedure (Renkl, 2002). From this point of view, our analysis suggests that both fourth-grade Taiwanese textbooks afford more opportunities for students to learn multiple aspects of addition and subtraction of fractions than those afforded by the other textbooks analyzed. As such, this finding reveals the importance of moving into more depth than simply examining the structural/organizational features of textbooks in order to explore how other textbook features (such as those included in the vertical analysis dimension of our framework) can structure opportunities for more complex and richer student learning.

Textbook Expectations for the Addition and Subtraction of Fractions

From the vertical dimension of the framework, we analyzed the potential cognitive demands of textbook tasks and the type of response required of students. Our interest was to discover what students would learn “if they had to solve all the exercises in the textbook” (Mesa, 2004, p. 256). Our findings suggested that tasks in the Taiwanese textbooks impose higher expectations on students compared to those of the textbooks from Cyprus and Ireland. Even tasks that required procedures without connections in the Taiwanese textbooks expected students to find missing addends; such tasks were rarely found in the Cypriot or Irish textbooks. Because substantially fewer tasks in Cypriot and Irish textbooks were of high cognitive demand and because research has shown that when setting up and implementing tasks, teachers are more likely to either maintain or reduce the cognitive demand of a task rather than to increase it (Stein et al., 2000, p. 25), students using the analyzed textbooks from Cyprus and Ireland are more likely to experience fewer tasks with a high cognitive demand than students in Taiwan. Including more cognitively demanding tasks in the Cypriot and Irish textbooks might not lead automatically to students’ engagement with tasks at a higher cognitive level, but it would set up a basis for teaching that takes advantage of those tasks.

Providing explanations is an important mathematics skill and one that relates to the practices of reasoning and communicating (Leinhardt, 2001; NCTM, 2000). In only one textbook (TTB) did we find tasks that required students to explain their solutions. Without exception, tasks in the Irish and Cypriot textbooks required students to simply supply the answer. Both Taiwanese textbooks expected more from students, namely writing a mathematical sentence related to a problem statement. Writing such sentences or providing explanations is important for helping students clarify their thinking and solution processes. Even though teachers in Cyprus and Ireland might ask students to provide such mathematical sentences of explanations when using the textbook with their students, the Taiwanese textbooks frequently reminded students (and teachers) to do this by building these expectations explicitly into the textbook tasks. By including higher level tasks and by asking their students to provide mathematical sentences and explanations, the Taiwanese textbooks, unlike the Cypriot and Irish textbooks, offered opportunities for their students to engage in increasingly complex mathematical challenges in the future.

The differences both with respect to the potential cognitive demand of the tasks and the type of responses required from the students seem to favor the Taiwanese textbooks. Based on this finding, one could conjecture that the high performance of East-Asian students in international comparative studies is related to differences in the textbooks used in these countries. We believe that such a claim would be unwarranted because it ignores the important mediating role of teachers in using curriculum materials in general, and textbooks, in particular (Remillard, 2005, 2009). At the same time, however, we should acknowledge that differences such as those identified in the present study suggest that textbooks themselves need to be considered when explaining the differences in instruction and consequently in students' performances in international comparative studies. For example, the June 2009 issue of the *ZDM—The International Journal on Mathematics Education* (Volume 41, Issue 3) highlights several features of quality instruction in East Asian countries, including a clear flow of the lesson and emphasis on key mathematical ideas, the use of several representations and solution strategies, the emphasis on explanations and reasoning, and the careful selection and sequencing of examples and tasks (Huang & Li, 2009; Kaur, 2009; Lin & Li, 2009; Pang, 2009). Although these instructional features (and consequently the effects they might have on student learning) are associated with teacher collaboration, teacher knowledge, the demands imposed on teachers, teaching-lesson competitions, and the continuous improvement of lessons in East Asian countries, the results of our study suggest that textbooks should be considered as well, because they may contribute to the quality of instruction and consequently to student learning.

Our analysis and discussion have so far focused on a specific topic: the treatment of the addition and subtraction of fractions in the textbooks of three countries. In the next section, we consider the contributions and the implications of this analysis from a broader perspective.

CONTRIBUTIONS AND IMPLICATIONS

In this section, we explore potential contributions of our study and its implications for textbook analysis. We also propose a theoretical hypothesis about the cultural character of textbooks. We conclude by raising questions that could motivate future research.

Textbook Analysis

Several components of our work in conducting this cross-national analysis and particularly in developing a framework that supports our analysis may be potentially valuable to future researchers. First, in our approach, we combined both a horizontal and a vertical analysis of textbooks; this approach yielded promising results, because it helped us identify disparate features of textbooks that might contribute to structuring student learning opportunities. We thus encourage future researchers to consider combining both types of textbook analysis. Second, we grounded the development of the framework in literature that specifically addressed student understanding of the content, in this sense using a top-down approach. Third, to be responsive to the differences across the three countries, we combined this with a bottom-up approach. In other words, we examined whether the criteria derived from the literature review sufficiently captured the characteristics of the textbooks in the three countries. Fourth, we explicitly documented challenges inherent in textbook analysis and how we resolved them. Below we discuss two such challenges.

The distinction between procedures without connections and procedures with connections with respect to the criterion of potential cognitive demands was difficult to establish because it involved a high level of inference. Even when closely following the *Task Analysis Guide* (see Stein et al., 2000, p. 16), which provides criteria for distinguishing between both types of tasks, we did not achieve satisfactory reliability. This difficulty stemmed from two sources, one associated with the *Task Analysis Guide* itself and the second associated with challenges in adapting this guide for textbook analysis. We believe there is a need to sharpen the distinction in the *Task Analysis Guide* between tasks that call on procedures with and without connections; at the same time, however, we acknowledge that the *Task Analysis Guide* was mainly developed to analyze demands of tasks as enacted during instruction. Without contextual detail (e.g., levels of prior instruction; other information accompanying each task, such as a representation), it is difficult to confidently assess the demands of a given textbook task. We resolved this difficulty in two ways. First, we simplified the coding by ignoring what students may have previously encountered; second, when we could not decide, we explicitly erred on the side of the higher level task. Although such an approach might be disputable, it allowed us to capture an important difference between the textbooks from the three countries.

Another challenge pertained to how we defined a task. Tasks, as numbered by textbook authors, varied substantially across textbooks and hence did not constitute a useful unit of analysis. Therefore, we defined tasks as “the questions asked in the exercises/problems” to make them more easily comparable across the three countries.

We opted to be explicit about such decisions for the benefit of other researchers. Such an approach can facilitate building on each other’s work (Schoenfeld, 1999).

Theoretical Implications

In their analysis of cross-national teaching, Stigler and Hiebert (1999) were “amazed at how much teaching varied across cultures and how little it varied within cultures” (p. 11). We found evidence that a similar phenomenon might be happening in textbooks produced within each individual country: the textbooks within Ireland and Taiwan—the countries for which multiple textbooks were available—appeared to be more similar to each other than to textbooks in the other country. For example, we found that the cognitive demands required of students tended to be higher in Taiwan than in Cyprus and Ireland. Fraction content was organized by the nature of the numbers in Cyprus and Taiwan but by operation in Ireland. More variety and combinations of fraction constructs were observed in Taiwan than in Cyprus, and Ireland had more tasks where no construct was evident. Worked examples were generally left incomplete in Taiwan, offered multiple solution approaches, and were accompanied by speech bubbles outlining possible solutions and students' thinking; in contrast, Irish worked examples, were always complete, offered only one solution approach and did not include such speech bubbles. In sum, our analysis suggested that more variation exists in the textbooks across countries rather than within the same country: textbooks published by different publishers within each country shared features that were not shared with textbooks in the other countries.

Our sample is limited to only one topic in two textbooks in each of the two countries where more than one textbook was in use. Still, the greater variability that existed across the two countries rather than within both countries suggested to us that choices made about textbooks may be cultural. One might attribute this difference in variability to differences in national curricula, especially given that the three countries examined here all had a mandated curriculum. But, curriculum alone—be it mandated or not—may not explain this difference in variability. We suggest that this dissimilar textbook variability across rather than within countries might be rooted in cultural differences, because the respective curricula do not specify features such as the nature of worked examples to be included in textbooks or the relative proportion of more or less cognitively demanding tasks.

Hiebert and colleagues (2003) used the term "lesson signature" to refer to distinctive patterns that can be observed among features of lessons taught by teachers in a particular country. Following the idea of "lesson signature," we wonder whether countries also have a "textbook signature." We define this signature as the uniform distinctive features in the textbooks within a particular country. This signature can be identified in and by textbook features such as those included in our framework. Although our analysis is limited to only two countries that had multiple textbooks, we conjecture that distinctive patterns may be observed among features of textbooks found within other countries; to the extent that this assumption holds, it would amount to a distinctive textbook signature in a country. If this textbook signature exists, it raises the question of its relationship to the "lesson signature" considered above: Does the "textbook signature" influence the teaching in a country and consequently the "lesson signature," or does the teaching in a country influence the "textbook signature," especially given that in many countries textbooks are written by experienced teachers? What are the nature and the mechanisms of these influences, should they exist? These questions alone further support the argument made earlier in the paper about the importance of textbooks in explaining differences in instruction and consequently students' performance and learning.

Implications for Future Studies

Our conjecture about a possible country-specific "textbook signature" warrants further investigation. For example, future studies could explore the degree to which this "textbook signature" exists in countries that do not have a national (mandated) curriculum; more research is needed to examine whether textbooks that are more or less widely used within a country share the country's "signature." Some additional questions also warrant investigation: Would other patterns emerge if we analyzed other topics from more textbooks using additional criteria from the framework, to reinforce our conjecture about a "textbook signature"? If "textbook signatures" exist, do they influence student learning? Looking for "textbook signatures" in other textbooks may help us understand more about variations in teaching across countries.

A related, broader, question pertains to the extent to which differences in textbooks matter for student learning. Because we did not study curriculum enactment, we cannot directly answer this question. Yet, given the prevalence of textbooks in schools, differences in textbooks may influence what happens in the classroom. Our multiple analyses enabled us to reveal differences that potentially affect both teaching and student learning. Future studies could explore if and how these differences play out in instruction and the extent to which teachers can capitalize on the affordances of the textbooks they use or, alternatively, help overcome textbook limitations.

Our findings also point to directions for future studies. One key finding related to our horizontal analysis was the difference in how topics were sequenced across textbooks in the three countries. Previous research (e.g., Resnick et al., 1989) suggests that the sequencing of the content affects student learning. The differences we found across textbooks prompt some questions: How are decisions made about the sequencing of topics in textbooks? To what extent are textbooks informed by available research findings? How do textbooks communicate the importance of sequence to teachers? These questions point to another genre of research that pertains to exploring the textbook authoring process itself and the extent and manner in which textbook authors across different counties capitalize on the available research findings.

Finally, alternative applications of the framework proposed here are also possible and may be investigated. For example, can the framework be used to study textbooks in other countries? What would be needed to apply the framework to topics other than the addition and subtraction of fractions? What other criteria might be usefully included in the framework? In thinking about these questions we offer the framework as an "open source," a basic set of tools that can be modified by users and adapted to address particular needs regarding textbook analyses, yet offering a common language that facilitates comparisons across those analyses.

One of the perspectives of cross-national textbook analysis we presented at the beginning of this paper views textbooks as affording probabilistic opportunities for what students in different countries learn when studying mathematics. The findings of our study resonate with this perspective, for they document specific ways in which textbooks can structure different learning opportunities for students. Both these findings and the directions for future studies listed above are suggestive of the richness of the field of cross-national textbook analysis and its potential for enhancing our understanding about what contributes to teaching and ultimately to student learning.

REFERENCES

Ball, D. L. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157–195). Hillsdale, NJ: Lawrence Erlbaum.

Barry, K., Manning, P., O'Neill, S. & Roche, T. (2003). *Mathemagic 5*. Dublin, Ireland: C.J. Fallon.

Baturo, A. R. (2004). Empowering Andrea to help year-5 students construct fraction understanding. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference for the Psychology of Mathematics Education* (vol. 2, pp. 95–102). Bergen: University College.

Beck, I., McKeown, M. G., & Gromoll, E. W. (1989). Learning from social studies texts. *Cognition and Instruction*, *6*, 99–158.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1993). Rational numbers: Toward a semantic analysis. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13–47). Hillsdale, NJ: Lawrence Erlbaum.

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–125). New York: Academic.

Boulet, G. (1998). Didactical implications of children's difficulties in learning the fraction concept. *Focus on Learning Problems in Mathematics*, *20* (4), 19–34.

Brousseau, G. (1997). *Theory and didactical situations in mathematics: Didactique des mathematiques 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds., and Trans.). Dordrecht, The Netherlands: Kluwer.

Carpenter, T. P., Coburn, T. G., Reys, R. E., & Wilson, J. W. (1976). Notes from national assessment: Addition and multiplication with fractions. *Arithmetic Teacher*, *23*, 137–141.

Carpenter, T. P., Fennema, E., & Romberg, T.A. (1993). Toward a unified discipline of scientific inquiry. In T. P. Carpenter, E. Fennema, & T.A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 1–11). Hillsdale, NJ: Lawrence Erlbaum.

Charalambous, C. Y., Delaney, S., Hsu, H., & Mesa, V. (2007). *Opportunities to learn fractions in fourth-grade elementary classes: Insights from an analysis of textbooks used in Cyprus, Ireland, and Taiwan*. Poster presented at the Annual Meeting of the American Educational Research Association. Chicago, Illinois.

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understanding of fractions. *Educational Studies in Mathematics*, *64*, 293–316.

Courtney, D. (2002). *Action maths: 5th class*. Dublin, Ireland: Folens.

Davis, J. D. (2007). Real-world contexts, multiple representations, student-invented terminology, and y-intercept. *Mathematical Thinking and Learning*, *9*, 387–418.

Delaney, S., Charalambous, Y. C., Hsu, H.-Y., & Mesa, V. (2007). The treatment of addition and subtraction of fractions in Cypriot, Irish, and Taiwanese textbooks. In J. H., Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 193–200). Seoul: The Korea Society of Educational Studies in Mathematics.

Dewey, J. (1906). *The child and the curriculum*. Chicago: The University of Chicago Press.

Freeman, D. J., & Porter, A. C. (1989). Do textbooks dictate the content of mathematics instruction in elementary schools? *American Educational Research Journal*, *26*, 403–421.

Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York: MacMillan.

Fuson, K. C., Stigler, J. W., & Bartsch, K. (1988). Grade placement of addition and subtraction topics in Japan, mainland China, the Soviet Union, Taiwan, and the United States. *Journal for Research in Mathematics Education*, *19*, 449–456.

Government of Ireland (1999). *Primary school curriculum: Mathematics*. Dublin, Ireland: The Stationery Office.

Haggarty, L., & Pepin, B. (2002). An investigation of mathematics textbooks and their use in English, French and German classrooms: Who gets an opportunity to learn what? *British Educational Research Journal*, *28*, 567–590.

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the "voice" of a mathematics textbook. *Journal for Research in Mathematics Education*, *38*, 344–369.

Herbst, P. (1995). *The construction of the real number system in textbooks: A contribution to the analysis of discursive practices in mathematics*. Unpublished master's thesis. University of Georgia, Athens.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J. K., et al. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study, (NCES 2003-013 Revised)*. Washington, DC: U.S. Department of Education, National Center for Education Statistics.

Howson, G. (1995). *Mathematics textbooks: A comparative study of grade-8 texts*. Vancouver, Canada: Pacific Education Press.

Huang, R., & Li., Y. (2009). Pursuing excellence in mathematics classroom instruction through exemplary lesson development in China: A case study. *ZDM—The International Journal on Mathematics Education*, *41*, 297–309.

Kaur, B. (2009). Characteristics of good mathematics teaching in Singapore grade-8 classrooms: A juxtaposition of teachers' practice and student perception. *ZDM—The International Journal on Mathematics Education*, *41*, 333–347.

Keijzer, R., & Terwel, J. (2001). Audrey's acquisition of fractions: A case study into the learning of formal mathematics. *Educational Studies in Mathematics*, *47*, 53–73.

Kieren, T. E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101–144). Columbus, OH: ERIC/SMEAC.

Lamon, S. J. (1999). *Teaching fractions and ratios for understanding*. Mahwah, NJ: Erlbaum.

Lamon, S. L. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *The roles of representations in school mathematics* (pp. 146–165). Reston, VA: National Council of Teachers of Mathematics.

Landis, J. R., & Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics*, *33*, 159–174.

Leinhardt, G. (2001). Instructional explanations: A commonplace for teaching and location for contrast. In V. Richardson (Ed.), *Handbook for research on teaching* (4th ed., pp. 333–357). Washington, DC: American Educational Research Association.

Li, K. (2004). *Mathematics 4 (vol. 1)*. Taipei, Taiwan: Han Lin.

Li, K. (2005). *Mathematics 4 (vol. 2)*. Taipei, Taiwan: Han Lin.

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. *Journal for Research in Mathematics Education*, *31*, 234–241.

Lin, P.-J., & Li, Y. (2009). Searching for good mathematics instruction at primary school level valued in Taiwan. *ZDM—The International Journal on Mathematics Education*, *41*, 363–378.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, *32*, 267–295.

Marshall, S. P. (1995). *Schemas in problem solving*. New York: Cambridge University Press.

Mayer, R. E., Sims, V., & Tajika, H. (1995). A comparison of how textbooks teach mathematical problem solving in Japan and the United States. *American Educational Research Journal*, *32*, 443–460.

McDonnell, L. M. (1995). Opportunity to learn as a research concept and a policy instrument. *Educational Evaluation and Policy Analysis*, *17*, 305–322.

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics*, *56*, 255–286.

Mesa, V. (2007, February). *Instructors and textbooks*. Paper presented at the 10th Conference of the SIG/MAA Research in Undergraduate Mathematics Education, San Diego, CA.

Mesa, V. (2010). Strategies for controlling the work in mathematics textbooks for introductory calculus. *Research in Collegiate Mathematics Education*, *16*, 235–265.

Ministry of Education (2000). *Grades 1–9 curriculum temporary guidelines*. Taipei, Taiwan: Author.

Ministry of Education and Culture (1998). *Mathematics for fourth grade (Vol. 1–4)*. Nicosia, Cyprus: Author.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education*, *30*, 122–147.

Mou, L., & Yang, J. (2004). *Mathematics 4 (vol. 1)*. Taipei, Taiwan: Kang Hsuan.

Mou, L., & Yang, J. (2005). *Mathematics 4 (vol. 2)*. Taipei, Taiwan: Kang Hsuan.

Mullis, I., Martin, M. O., Gonzalez, E. J., & Chrostowski, S. J. (2004). *TIMSS 2003 International mathematics report*. Chestnut Hill, MA: Lynch School of Education & Boston College.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Newton, D. P., & Newton, L. D. (2006). Could elementary mathematics textbooks help give attention to reasons in the classroom? *Educational Studies in Mathematics*, *64*, 69–84.

Pang, J. S. (2009). Good mathematics instruction in South Korea. *ZDM—The International Journal on Mathematics Education*, *41*, 349–362.

Patton, M. Q. (2002). *Qualitative research and evaluation methods* (3rd ed.). Thousand Oaks, CA: Sage.

Pepin, B., Haggarty, L., & Keynes, M. (2001). Mathematics textbooks and their use in English, French and German classrooms: A way to understand teaching and learning cultures. *ZDM Mathematics Education*, *33*, 158–174.

Raths, L. E. (1932). The grade-placement of addition and subtraction of fractions. *Educational Research Bulletin*, *11* (2), 29–38.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*, 211–246.

Remillard, J. T. (2009). Part II-Commentary: Considering what we know about the relationship between teachers and curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 85–92). New York: Routledge.

Renkl, A. (2002). Worked-out examples: Instructional explanations support learning by self-explanations. *Learning and Instruction*, *12*, 529–556.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetical errors: The case of decimal fractions. *Journal for Research in Mathematics Education*, *20*, 8–27.

Rezat, S. (2006). A model of textbook use. In J. Novotnå, H. Moraovå, M. Kråtkå, & N. Stehlíkovå (Eds.), *Proceedings of the 30th Conference of the international group of the Psychology of Mathematics Education* (vol. 4, pp. 409–416). Prague: Charles University.

Schmidt, W. H., McKnight, C. C., Valverde, G., Houang, R. T., & Wiley, D. E. (1997). *Many visions, many aims: A cross-national investigation of curricular intentions in school mathematics*. Dordrecht, The Netherlands: Kluwer.

Schoenfeld, A. H. (1999). Looking toward the 21st Century: Challenges of educational theory and practice. *Educational Researcher*, *28*, 4–14.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36.

Smith, J. P. (2002). The development of students' knowledge of fractions and ratios. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 3–17). Reston, VA: National Council of Teachers of Mathematics.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*, 455–488.

Stein, M. K., Smith, M. S., Henningsen, M., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction*. New York: Teachers College Press.

Stevenson, H. W., & Bartsch, K. (1992). An analysis of Japanese and American textbooks in mathematics. In R. Leetsma & H. Walberg (Eds.), *Japanese educational productivity* (pp. 103–133). Ann Arbor: Center for Japanese Studies, University of Michigan.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap*. New York: Free Press.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage.

Tate, W. F., & Rousseau, C. (2006). Engineering change in mathematics education. In F. K. Lester (Ed.), *Second handbook for research in mathematics teaching and learning* (pp. 1209–1246). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards of School Mathematics* (pp. 95–113). Reston, VA: National Council of Teachers of Mathematics.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice in the world of textbooks*. Dordrecht, The Netherlands: Kluwer.

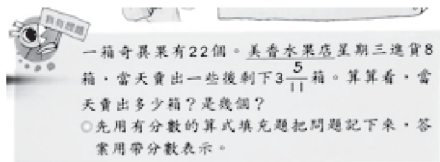
Verschaffel, G., Greer, B., & Torbeyns, J. (2006). Numerical thinking. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present, and future* (pp. 51–82). Rotterdam, The Netherlands: Sense Publishers.

Vinner, S., Hershkowitz, R., & Bruckheimer, M. (1981). Some cognitive factors as causes of mistakes in the addition of fractions. *Journal for Research in Mathematics Education*, *12*, 70–76.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Lawrence Erlbaum.

APPENDIX

Examples Illustrating the Coding of the Tasks



One box has 22 kiwis. May-Chin's store bought 8 boxes on Wednesday and sold out some of them. Now she has 3 5/11 boxes left. How many boxes of kiwis did May-Chin sell on Wednesday? This is equal to how many kiwis? Write a mathematical expression and write your answer as a mixed fraction. (TTB, vol. 2, p. 23)

Construct: Part-whole and operator; Representations: Decorative; Cognitive demands: Procedures with connections; Performance expectations: Answer and mathematical sentence.

A box contains 8/10 kg of flour. Mrs. Georgia used 5/10 of the flour. What part of the flour is left? (CT, vol. 3, p. 45)
Construct: Measure; Representations: None; Cognitive demands: Procedures with connections; Performance Expectations: Answer only.

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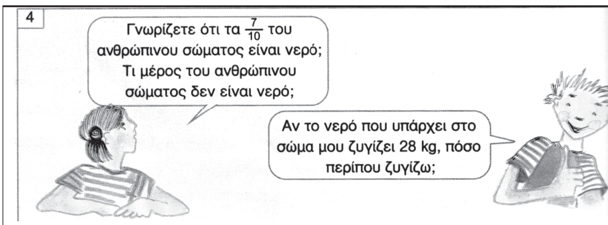
8. (a) $\frac{1}{3}+\frac{2}{12} \Rightarrow \frac{\square}{12}+\frac{\square}{12} \Rightarrow \square = \square$ (b) $\frac{1}{6}+\frac{1}{12} \Rightarrow \frac{\square}{12}+\frac{\square}{12} \Rightarrow \square = \square$

(c) $\frac{1}{3}+\frac{5}{12} \Rightarrow \frac{\square}{12}+\frac{\square}{12} \Rightarrow \square = \square$ (d) $\frac{1}{4}+\frac{5}{12} \Rightarrow \frac{\square}{12}+\frac{\square}{12} \Rightarrow \square = \square$

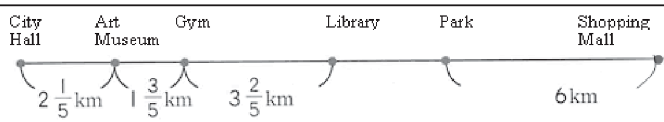
ITA, p. 47

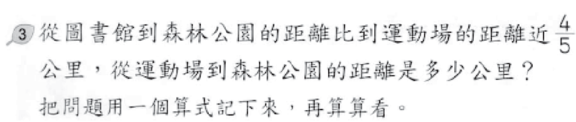
Construct: None; Representations: None; Cognitive demands: Procedures without connections; Performance Expectations: Answer only.



Girl: Did you know that 7/10 of the human body consists of water? What part of the human body does not consist of water?
Boy: If the water in my body weighs 28 kg, how much do I weigh?
(CT, vol. 3, p. 46)

This problem has two tasks (one for the girl's question; another for the boy's question).
First task: Construct: Part-whole; Representations: Relevant to the context but not to the mathematics; Cognitive demands: Procedures with connections; Performance Expectations: Answer only.
Second task: Construct: Part-whole; Representations: Relevant to the context but not to the mathematics; Cognitive demands: Doing mathematics (the student is given the part and is expected to find the whole); Performance Expectations: Answer only.





The distance between the Library and the Park is 4/5km shorter than that between the Library and the Gym. How many kilometers is the distance between the Gym and the Park? Transfer this problem into a mathematical sentence and find the answer. (TTA, vol. 2, pp. 86-87)

Construct: Measure; Representations: Linear; Cognitive demands: Doing mathematics; Performance Expectations: Answer and mathematical sentence.