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History of Research in Mathematics Education

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Keywords

Research; Mathematics education; History; Academic field; Psychology; Mathematics

Definition

An account of activities and events concerned with the development of disciplined inquiry in mathematics education as a flourishing academic enterprise.

Main Text

Although mathematics has been taught and learned for millennia, not until the past century or so have the nature and quality of teaching and learning mathematics been studied in any a serious manner. Clay tablets from ancient Babylonia (c 1900 BC to c 1600 BC), for example, show that students in the scribal school were expected to solve problems involving quadratic polynomials (Høyrup 1994, pp. 4–9), but no available evidence indicates how much drill and practice either they received or their instructors thought they needed. As of 1115 BC, applicants to the Chinese civil service had to pass an examination in arithmetic (Kilpatrick 1993, p. 22), but as far as anyone knows, no one ever investigated how well their examination performance predicted their job performance. In Plato's *Meno*, he relates how, in the fifth century BC, Socrates helped a slave boy discover that doubling the side of a square apparently squares its area. Plato does not, however, say how well the boy fared with similar geometry problems once his teacher was no longer around. Mathematics education is a long-established field of practice; research in mathematics education, a relatively recent enterprise.

Over the centuries, teachers of mathematics in various countries have offered reflective accounts of their work, often writing textbooks constructed around teaching techniques they developed out of their own experience. Only during the nineteenth century, however, as national educational systems were established and the training of teachers moved into colleges and universities, did people begin to identify themselves as mathematics educators and begin to conduct research as part of their scholarly identity (Kilpatrick 1992, 2008). Not until 1906 were the first doctorates in mathematics education granted – to Lambert L. Jackson and Alva W. Stamper, students of David Eugene Smith at Teachers College, Columbia University (Donoghue 2001). Within the next few decades, research in mathematics education gradually began to be conducted in several countries as lectures in mathematics education were offered and

graduate programs in mathematics education became established in universities.

Mathematics Education as an Academic Field

The education of teachers, which had often been a hit-or-miss affair, did not become a field of professional studies until the nineteenth century. Although teacher-training schools had begun in France and Prussia late in the seventeenth century, only in the eighteenth century were normal schools – very much influenced by the ideas of the Swiss pedagogue and reformer Johann H. Pestalozzi – established in European countries (Cubberley 1919). In 1829, the American geographer William C. Woodbridge, who in the previous 4 years in Europe had observed schools in Prussia and Switzerland and had visited Pestalozzi, tried unsuccessfully to establish in Hartford, Connecticut, a teachers seminary modeled after the Prussian version. In 1831, he observed: "In those of the countries of Europe where education has taken its rank as a science, it is almost as singular to question the importance of a preparatory seminary for teachers, as of a medical school for physicians" (quoted by Cubberley 1919, p. 374). Education in general had slowly been entering the university since the eighteenth century, beginning with a chair of education established at the University of Halle in 1779, but not until the late nineteenth and early twentieth centuries were such chairs established elsewhere, and only then did school mathematics start to become an object of scholarly study (Kilpatrick 2008).

Many of the early researchers in mathematics education were mathematicians who had become interested in how mathematics is done. For example, the editors of *L'Enseignement Mathé matique*, Henri Fehr and Charles-Ange Laisant, sent a questionnaire to over 100 mathematicians to learn how they did mathematics. The report of their survey, which was published in 11 installments in the journal from 1905 to 1908, was essentially a list of verbatim responses to their questions. In contrast, the French mathematician Jacques Hadamard later undertook a similar but less formal inquiry into the working habits of mathematicians in America that went somewhat deeper into the methods and images they used (Kilpatrick 1992). Other early researchers were psychologists who were developing an interest in how children think about and learn mathematical ideas. Beginning in 1875, with Wilhelm Wundt's establishment of a laboratory in Leipzig and William James's establishment of one at Harvard, dozens of psychological laboratories were established in Europe, Asia, and North America (Kilpatrick 1992). Psychologists such as Alfred Binet, his colleague Jean Piaget, Max Wertheimer, Otto Selz, and Lev Vygotsky investigated mental ability and productive thinking using mathematical tasks. Psychology was becoming the so-called master science of the school: "Psychology . . . became the guiding science of the school, and imparting to would-be teachers the methodology of instruction, in the different school subjects, the great work of the normal school" (Cubberley 1919, p. 400). Together, mathematicians and psychologists began the efforts that would lead to research in mathematics education.

Comparative Studies of School Mathematics

In 1908, the International Commission on the Teaching of Mathematics (ICTM) was formed at the Fourth International Congress of Mathematicians in Rome. Its purpose was "to report on the state of mathematics teaching at all levels of schooling around the world" (Kilpatrick 1992, p. 6). In 1912, at the Fifth International Congress in Cambridge, England, some 17 countries presented reports, and by 1920, the countries active in the ICTM had produced almost 300 reports (Schubring 1988; Furinghetti 2008). The international comparisons based on these reports, however, were essentially restricted to descriptions by a handful of mathematicians or educators in each country of activities that they were aware of. They did not engage in large-scale, systematic surveys of the

school mathematics curriculum, nor did they visit classrooms to record instructional practices. Nonetheless, they had begun the process of looking across countries to get a better perspective on mathematics education around the world.

In the last half century, researchers have undertaken a variety of international comparative assessments of students' mathematical knowledge and of teachers' knowledge of pedagogy and mathematics. They have also compared mathematics teaching across countries using video records of lessons. (For an analysis of the levels at which these comparisons have been made, see Artigue and Winsløw 2010). Considerable progress has been made in both the thoroughness with which such comparative studies have been done and the sophistication of the data collection and analyses. Although these studies can be criticized for being too oriented toward Western practice and inadequately sensitive to Asia-Pacific cultures (Clements and Ellerton 1996), they have had, in many countries, considerable influence on curriculum, teaching, and educational policy. For an account of the development of international collaboration in mathematics education during the past century, see Karp (2013).

Becoming Scientific

In trying to make their field scientific, educational psychologists looked to the natural sciences for models, and in much the same way, some mathematics educators seeking to establish their field as a science took those sciences as models. They studied mathematics learning under controlled laboratory conditions, testing hypotheses about the effects of various "treatments," and making careful measurements of the learning achieved. Influential examples were studies by the psychologist Edward L. Thorndike in the early years of the twentieth century. Using a control group whose performance was compared with that of an experimental group (with students assigned randomly to one of the two groups), Thorndike demonstrated that practice by the experimental group in performing certain tasks such as judging the size of rectangles did not improve their performance in – that is, did not transfer to – judging the size of triangles (Kilpatrick 1992). Thorndike's research studies dealt a major blow to arguments that mathematics ought to be taught and learned because the logical thinking it promoted transferred to other realms. He argued that his research showed that transfer was much more limited than mathematics teachers appeared to assume.

Thorndike not only published important books on the psychology of arithmetic and the psychology of algebra in which he promoted the psychology he termed connectionism; he also published a series of arithmetic textbooks that was widely used in schools. Connectionism became the forerunner of the behaviorism that came to dominate much of research in mathematics education in the United States from the 1930s through the 1950s (Clements and Ellerton 1996). Although other psychologists, such as Charles H. Judd, Guy T. Buswell, and William A. Brownell, performed research studies that called Thorndike's work into question, thereby developing a psychology of the school subjects that mathematics educators found more congenial (Kilpatrick 1992), connectionism and its successor behaviorism exerted a much stronger influence on research methodology in mathematics education for many years and not just in the United States.

Elsewhere in the first decade of the twentieth century, some psychologists were looking at errors and difficulties that children were having in arithmetic. Paul Ranschburg in Budapest, in particular, began the study of differences in calculation performance between normal children and low achievers in arithmetic. In 1916, he coined the term *Rechenschwäche* (dyscalculia) for severe inability to perform simple arithmetic calculations (Schubring 2012). Like Thorndike, Ranschburg attributed children's successful performance to their possession of *Vorstellungsketten* (chains of association), but his research method relied more on observation of differences between existing groups (normal and low achieving) than on experimentation.

Psychologists gradually stopped being so concerned about emulating the natural sciences

and began to develop their own techniques for studying learning, and researchers in mathematics education followed. For example, in the movement known as "child study" (Kilpatrick 1992), which had appeared in Germany and the United States at the end of the nineteenth century, researchers looked at the development of concepts in young children using techniques of observation and interview. Although mathematics was not often the focus of child study research, it did give rise to a number of descriptive, naturalistic studies. Less than a century later, research on the learning of mathematics had burgeoned. A survey in the 1970s, for example, located some 3,000 published studies of mathematics learning (Bauersfeld 1979).

Studying the Teaching of Mathematics

As mathematics educators began to study children's mathematics learning and thinking, they increasingly recognized that laboratory studies present a restricted view of those processes; however, they are conceived. Children do most of their learning of mathematics in school classrooms along with other children, and their thinking about mathematical concepts and problems is much influenced by others, including their teacher. The psychologist Ernst Meumann, who had studied with Wundt in Leipzig, was one of the first to address what he called "experimental pedagogy" and in 1914 published a volume in which he looked at the didactics of teaching specific school subjects (Schubring 2012). Meumann was the forerunner of researchers who were later in the century to establish a critically important field of research, especially in Germany and France: the *didactics of mathematics* (Artigue and Perrin-Glorian 1991; Biehler et al. 1993). Although the didactics of mathematics began with a psychological orientation, it came under the influence of other fields – anthropology and philosophy, in particular – as it was increasingly located in university departments of mathematics and began to become established as one of the mathematical sciences.

Didactics of mathematics, however, was not the only research effort to address mathematics teaching. In a number of studies conducted in the first half of the twentieth century, components of teaching or characteristics of teachers were linked to learners' performance in efforts to understand what might constitute effective teaching. Researchers eventually moved from such simple "process-product" models to more sophisticated efforts that attempted to capture more of the complexity of the teaching-learning process, including the knowledge and beliefs of the participants as well as their activities during instruction. For an account of the gradual elaboration of research models for studying mathematics teaching, see Koehler and Grouws (1992).

In later developments, researchers attempted to go deeper into questions of what constitutes classroom practice in mathematics and how that is experienced by teachers and learners. In particular, they studied how discourse is structured in mathematics classes, how norms are established in classrooms for learning and doing mathematics, and how teachers and learners build relationships based on getting to know each other (Franke et al. 2007). Research on teaching and teachers has become a major strand of current research in mathematics education, and those studies now extend from preschool to tertiary instruction.

An especially fertile development of recent decades has been the growth of research on technology and digital environments for mathematics teaching and learning. Physical tools have been used for centuries to assist the teaching and learning of mathematics, and an examination of how those tools have been used can help put into perspective the use of computing technology today (Roberts et al. 2013). In an early review of how electronic technologies had been studied in mathematics education research, Kaput and Thompson (1994) lamented the paucity of technology-related research publications. That situation has changed dramatically since that review, as numerous recent books (e.g., Guin et al. 2005; Hoyles and Lagrange 2010) and journals (e.g., *International Journal for*

Technology in Mathematics Education; *Journal of Computers in Mathematics and* Science Teaching) attest.

A Flourishing Academic Enterprise

The last half century has witnessed a growing flood of research activity in mathematics education that has been an integral part of its growth and development:

> Today an astonishing profusion of books, handbooks, proceedings, articles, research reports, newsletters, journals, meetings, and organizations is devoted to mathematics education in all its aspects. A search of the scholarly literature on the Web for the phrase *mathematics education* yields 125,000 hits; a search of the entire Web yields almost 9 times that number. (Kilpatrick 2008, p. 38).

One measure of the maturation of the field of mathematics education is that researchers have begun to study its history. A major milestone was the founding in 2006 of the *International Journal for the History of Mathematics Education*. The history of the field had been discussed at various international conferences beginning in 2004, and a series of biennial conferences devoted to the topic began in Iceland in 2009.

As the field of mathematics education has grown, research in the field has grown even faster. The subject matter of research studies has broadened to include such topics as the school mathematics curriculum, assessment in mathematics, the education of mathematics teachers and their professional development, the sociopolitical context of learning and teaching mathematics, teaching mathematics to students in special education programs, and the politics of mathematics education. The methods used to conduct research now go well beyond experimentation to include case studies of teachers and students, surveys of attitudes and beliefs, and ethnographies of cultural practices.

Organizations of researchers have been formed that range from those of international scope, such as the International Group for the Psychology of Mathematics Education (IGPME, or PME), to organizations within one or several countries, such as the Canadian Mathematics Education Study Group (CMESG), the French *Association pour la Recherche en Didactique des Mathématiques* (ARDM), and the Mathematics Education Research Group of Australasia (MERGA). For a comprehensive survey of international or multinational organizations in mathematics education, see Hodgson et al. (2013). Many of these organizations hold regular conferences on research and publish research journals. Mainstream journals that have been publishing research for more than four decades, such as *Educational Studies in Mathematics* and the *Journal for Research in Mathematics Education*, have lately been joined by more specialized research journals such as the *Journal of Mathematics and Culture*, started in 2006, and the *Journal of Urban Mathematics Education*, started in 2008. For an account of the growth of journals and research conferences in mathematics education, see Furinghetti et al. (2013). The sheer volume of research activity being reported in these journals and at these conferences is staggering. A comprehensive portrayal of research activity in mathematics education today is no longer possible; the terrain is simply too extensive and diverse to be captured in toto.

Cross-References

- ▶ Anthropological Approaches in Mathematics Education, French Perspectives
- ▶ Constructivist Teaching Experiment
- ▶ Design Research in Mathematics Education
- ▶ History of Mathematics Teaching and Learning
- ▶ International Comparative Studies in Mathematics: An Overview
- ▶ Teacher as Researcher in Mathematics Education
- ▶ Theories of Learning Mathematics

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Hypothetical Learning Trajectories in Mathematics Education

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Keywords

Learning; Teaching; Constructivism; Teacher thinking; Learning progressions

Definition

Hypothetical learning trajectory is a theoretical model for the design of mathematics instruction.