Γενικά υπάρχει ένας πίνακας $A \in \mathbb{R}^{n \times n}$ κ' $b \in \mathbb{R}^n$ τέτοιο $x \in \mathbb{R}^n$ $A\underline{x} = \underline{b}$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$A \in \mathbb{R}^{n \times n}$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \leftarrow r(A) = n$$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \leftarrow r(A|b) = \min(n, n+1)$$

Gauss έχουμε οδηγό στοιχείο το $a_{11}$ και βρίσκουμε στην κύρια διαγώνιο να μηδενίσω τα στοιχεία $a_{21}, \ldots, a_{n1}$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \qquad \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & b_2^{(1)} \\ 0 & a_{32}^{(1)} & \cdots & a_{3n} & b_3 \\ \vdots & & & & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & b_n^{(1)} \end{array}\right)$$

$$a_{21}^{(1)} = 0 = a_{21} + \frac{-a_{21}}{a_{11}} a_{11} \qquad l_{21} = -\frac{a_{21}}{a_{11}}$$

$$a_{22}^{(1)} = a_{22} + l_{21} \cdot a_{12} \qquad b_2^{(1)} = b_2 + l_{21} \cdot b_1$$

$$a_{22}^{(1)} = a_{22} + l_{21} \cdot a_{12} \qquad b_2^{(1)} = b_2 + l_{21} \cdot b_1$$

$$a_{23}^{(1)} = a_{23} + l_{21}\, a_{13}$$

$$\vdots$$

$$a_{2n}^{(1)} = a_{2n} + l_{21} \cdot a_{1n}$$

$$a_{31}^{(1)} = 0 = a_{31} + \frac{-a_{31}}{a_{11}}\, a_{11} \qquad l_{31} = -\frac{a_{31}}{a_{11}}$$

$$a_{32}^{(1)} = a_{32} + l_{31} \cdot a_{12} \qquad b_3^{(1)} = b_3 + l_{31} \cdot b_1$$

$$\vdots$$

$$a_{3n}^{(1)} = a_{3n} + l_{31} \cdot a_{1n}$$

$$a_{ij}^{(1)} = a_{ij} + l_{i1} \cdot a_{1j} \qquad j = 2, 3, \dots, n$$

$$b_i^{(1)} = b_i + l_{i1} \cdot b_1 \qquad l_{i1} = -\frac{a_{i1}}{a_{11}} \quad i = 2, \dots, n$$

$$\left(\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\
0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} & b_2^{(1)} \\
0 & a_{32}^{(1)} & a_{33}^{(1)} & \dots & a_{3n}^{(1)} & b_3^{(1)} \\
\vdots & & & & & \vdots \\
0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \dots & a_{nn}^{(1)} & b_n^{(1)}
\end{array}\right)$$

$$\left(\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\
0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} & b_2^{(1)} \\
0 & 0 & a_{33}^{(2)} & \dots & a_{3n}^{(2)} & b_3^{(2)} \\
\vdots & \vdots & \vdots & & & \vdots \\
0 & 0 & a_{n3}^{(2)} & \dots & a_{nn}^{(2)} & b_n^{(2)}
\end{array}\right)$$

$$a_{32}^{(2)} = 0$$

$a_{32}^{(2)} = 0$

$a_{42}^{(2)} = 0$ ..

$a_{n2}^{(2)} = 0$

$a_{32}^{(2)} = 0 = a_{32}^{(1)} + \frac{-a_{32}^{(1)}}{a_{22}^{(1)}} a_{22}^{(1)}$ $\qquad l_{32} = -\frac{a_{32}^{(1)}}{a_{22}^{(1)}}$

$a_{33}^{(2)} = a_{33}^{(1)} + l_{32} a_{23}^{(1)}$

$a_{3n}^{(2)} = a_{3n}^{(1)} + l_{32} a_{2n}^{(1)}$

$b_3^{(2)} = b_3^{(1)} + l_{32} b_2^{(1)}$

$$\left(\begin{array}{cccccc|c}
a_{11} & a_{12} & a_{13} & \cdots & & a_{1n} & b_1 \\
0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & & a_{2n}^{(1)} & b_2^{(1)} \\
0 & 0 & a_{33}^{(2)} & \cdots & & a_{3n}^{(2)} & b_3^{(2)} \\
\vdots & \vdots & 0 & a_{44}^{(3)} & \cdots & & \\
\vdots & \vdots & \vdots & 0 & \ddots & & \\
0 & 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} & b_n^{(n-1)}
\end{array}\right)$$

- Το άνω τριγωνικό κομμάτι που προκύπτει από τον $A$ με την μέθοδο Gauss ο συμβολίζουμε $U$ κ' το νέο δεξί μέλος από το $b \to \tilde{b}$. Δηλαδή από $A|b \to U|\tilde{b}$
- Στο μετασχηματισμένο σύστημα $Ux = \tilde{b}$ η τελευταία εξίσωση έχει μόνο έναν άγνωστο, το $x_n$, και αυτό βρίσκουμε με την επίλυση $a_{nn}^{(n-1)} x_n = b_n^{(n-1)}$

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}, \quad a_{nn}^{(n-1)} \neq 0$$

- Σε συνέχεια αντικαθιστούμε το $x_n$ στην $n-1$ εξίσωση όπου λύνουμε ως προς το άγνωστο $x_{n-1}$
- Με την ίδια λογική → έως το $x_1$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \ddots & a_{n-1,n-1} \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

$$a_{nn} x_n = b_n \Leftrightarrow x_n = \frac{b_n}{a_{nn}} \qquad a_{nn} \neq 0$$

$$a_{n-1,n-1} \underset{\text{ή}}{x_{n-1}} + a_{nn} x_n$$

$$0x_1 + 0x_2 + 0x_3 + \cdots + a_{n-1,n-1} x_{n-1} + a_{n,n} x_n = b_{n-1} \Leftrightarrow$$

$$x_{n-1} = \frac{b_{n-1} - a_{n,n} x_n}{a_{n-1,n-1}} \quad \cdots\cdots \; x_1 = \cdots\cdots$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & & \vdots \\ l_{31} & l_{32} & 1 & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ l_{n1} & \cdots & & l_{n,n-1} & 1 \end{pmatrix} = L$$

$\underline{x}$ διάνυσμα που ψάχνω

$$L \cdot U = A$$

$$A\underline{x} = \underline{b} \qquad L\underline{y} = \underline{b} \quad 1$$

$$\left. L \underbrace{U\underline{x}}_{y} = \underline{b} \right\} \qquad U\underline{x} = \underline{y} \quad 2$$

$\cdot n$

$$\underbrace{L\,U\underline{x}}_{\underline{y}} = \underline{b} \quad ; \quad U\underline{x} = \underline{y} \quad 2$$

Παράδειγμα

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 3x_1 + 2x_2 + x_3 &= 2 \\ -2x_1 + 3x_2 + 5x_3 &= 0 \end{aligned}\right\} \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ -2 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ -2 & 3 & 5 & 0 \end{array}\right) \begin{array}{l} r_2' = r_2 - 3r_1 \\ r_3' = r_3 + 2r_1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 5 & 7 & 2 \end{array}\right) r_2'' = (-1)\cdot r_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 7 & 2 \end{array}\right) r_3''' = r_3'' - 5r_2'' \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right) r_3'''' = (-1)r_3'''$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right) \to \underset{3\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{3\times 1}{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}$$

$$* \quad 3x_3 = 3 \Leftrightarrow x_3 = 1$$

$$x_2 + 2x_3 = 1 \Leftrightarrow x_2 = 1 - 2x_3 = 1 - 2\cdot 1 = -1$$

$$x_1 + x_2 + x_3 = 1 \Leftrightarrow x_1 = 1 - x_2 - x_3 = 1 - (-1) - 1 = 1$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad * \left.\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 0x_1 + x_2 + 2x_3 &= 1 \\ 0x_1 + 0x_2 + 3x_3 &= 3 \end{aligned}\right\} \quad 3x_3 = 3 \Rightarrow x_3 = 1$$

Παράδειγμα

$$\left.\begin{aligned} x_1 - x_2 + 0x_3 &= 0 \\ 2x_1 + 0x_2 + x_3 &= 3 \end{aligned}\right\} \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

$$2x_1 + 0x_2 + x_3 = 3$$
$$x_1 - 2x_2 + 0x_3 = -1$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} \\ x_3 \end{pmatrix} \begin{pmatrix} \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \end{array}\right) \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - r_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \end{array}\right) r_2'' = r_2' \cdot \frac{1}{2}$$

$$b = 0 + \left(\frac{-b}{a}\right) a$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -1 & 0 & -1 \end{array}\right) r_3''' = r_3'' + r_2'' \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right) r_3'''' = 2r_3'''$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{array}\right) r_2' = r_2 - \frac{1}{2} r_3 \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) r_1'' = r_1' + r_2'$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x_1 + 0x_2 + 0x_3 = 1 \Rightarrow x_1 = 1$$
$$0x_1 + x_2 + 0x_3 = 1 \Rightarrow x_2 = 1$$
$$0x_1 + 0x_2 + x_3 = 1 \Rightarrow x_3 = 1$$

$(x_1 = 1 \cdot x_1 \,,\ -x_1 = -1 \cdot x_1)$

Παράδειγμα

$$x_1 - x_2 + x_3 = 2$$
$$5x_1 - 2x_2 + 4x_3 = 13$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 5 & -2 & 4 \\ \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ 13 \\ a \end{pmatrix}$$

$$5x_1 - 2x_2 + 4x_3 = 13$$
$$3x_1 + 0x_2 + 2x_3 = 9$$

$$\begin{pmatrix} 5 & -2 & 4 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 5 & -2 & 4 & 13 \\ 3 & 0 & 2 & 9 \end{array}\right) \begin{array}{l} r_2' = r_2 - 5r_1 \\ r_3' = r_3 - 3r_1 \end{array} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 3 & -1 & 3 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Γραμμική εξάρτηση διανυσμάτων

$\det(A) = 0$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \Leftrightarrow$$

$$x_1 - x_2 + x_3 = 2$$
$$3x_2 - x_3 = 3$$
$$0x_1 + 0x_2 + 0x_3 = 0$$

- Έχω άπειρες λύσεις
- Θέτω $x_3 = k \quad k \in \mathbb{R} \quad x_2 = (3 + x_3)/3 = 1 + \frac{x_3}{3} = 1 + \frac{k}{3}$

$$x_1 = 2 + x_2 - x_3 = 2 + 1 + \frac{k}{3} - k = 3 - \frac{2k}{3} \quad k \in \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 - \frac{2k}{3} \\ 1 + \frac{k}{3} \\ k \end{pmatrix} \quad k \in \mathbb{R} \quad = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} k$$

$A \in \mathbb{R}^{n \times n} \quad r(A) < n \quad$ κ' $r(A) = k$

$(r(A) = 2)$

$n - k$ μηδενικές γραμμές. $\quad (n - 2 = 3 - 2 = 1)$

$n-k$ μηδενικές γραμμές. $(n-2-\dots)$

Άρα μπορώ να ορίσω $n-k$ μεταβλητές (έστω $\kappa, \in \mathbb{R}$)
και να εκφράσω τα $x_1, x_2, \dots, x_{n-(k-1)}$ στοιχεία σε
συνάρτηση των $\underbrace{\kappa, \lambda, \dots}_{n-k \text{ σε πλήθος}}$.

Δηλαδή τα $x_{k+1}, x_{k+2}, \dots, x_n$ θα εκφραστούν με μεταβλητές.

Έστω ότι είχα $A_{10\times 10} \rightarrow r(A) = 6$ δηλαδή
θα έχει 4 μηδενικές γραμμές.
Θα έχω δηλαδή 10 αγνώστους & 6 εξισώσεις.
Θα εκφράσω τα $x_7, x_8, x_9, x_{10}$ με $\kappa, \lambda, \mu, \nu \in \mathbb{R}$.
οπότε τα υπόλοιπα $x_1, x_2, x_3, x_4, x_5, x_6$ θα εκφραστούν
σε συνάρτηση αυτών $x_7, x_8, x_9, x_{10}$ που έχουν πάρει
τιμές από τα $\kappa, \lambda, \mu, \nu$.

<u>Παράδειγμα.</u>

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 0 \\ 2x_1 + 5x_2 - 3x_3 &= 1 \\ x_1 + 5x_2 + x_3 &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 2 & -2 \\ 2 & 5 & -3 \\ 1 & 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 2 & 5 & -3 & 1 \\ 1 & 5 & 1 & 1 \end{array}\right) \begin{array}{l} \Gamma_2' = \Gamma_2 - 2\Gamma_1 \\ \Gamma_3' = \Gamma_3 - \Gamma_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 1 \end{array}\right) \Gamma_3'' = \Gamma_3' - 3\Gamma_2'$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 1 \\ \dots \end{array}\right) \text{Συν.} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ \dots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \end{array}\right) \quad \text{Σημ.} \quad \underbrace{\begin{pmatrix} [illegible] & [illegible] & [illegible] \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} [illegible] \\ x_2 \\ x_3 \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} [illegible] \\ 1 \\ -2 \end{pmatrix}}_{\underline{b}}$$

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 0 \\ + x_2 + x_3 &= 1 \\ + 0x_3 &= -2 \quad !!! \end{aligned}$$

<u>αδύνατο</u>

Το σύστημα δεν έχει λύση.

Μη τετραγωνικοί πίνακες

$A \in \mathbb{R}^{m\times n}$ $\quad m > n \quad r(A) = \min(m,n) = n$

- $r(A|b) = r(A)$ τότε το σύστημα έχει λύση
  - $r(A) = r(A|b) = n$ η λύση μοναδική
  - $r(A|b) = r(A) < n$ έχει άπειρες λύσεις
- $r(A|b) > r(A)$ το σύστημα δεν έχει λύση

Παράδειγμα.

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 1 \\ x_1 - 2x_2 + 3x_3 + 2x_4 &= 9 \\ 2x_1 + x_2 - x_3 + 0x_4 &= 4 \\ x_1 + 0x_2 + 0x_3 - x_4 &= -1 \\ 0x_1 + x_2 - x_3 + 2x_4 &= 6 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 1 & -2 & 3 & 2 & 9 \\ 2 & 1 & -1 & 0 & 4 \\ 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 2 & 6 \end{array}\right) \begin{array}{l} \\ r_2' = r_2 - r_1 \\ r_3' = r_3 - 2r_1 \\ r_4' = r_4 - r_1 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \end{array}\right.$$

$0x_1 + x_2 - x_3 + 2x_4 = 6$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & -3 & 2 & 3 & 8 \\ 0 & -1 & -3 & 2 & 2 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & 1 & -1 & 2 & 6 \end{array}\right) \xrightarrow{r_2'' \leftrightarrow r_5''} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & -1 & -3 & 2 & 2 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & -3 & 2 & 3 & 8 \end{array}\right) \begin{array}{l} r_3''' = r_3'' + r_2'' \\ r_4''' = r_4'' + r_2'' \\ r_5''' = r_5'' + 3r_2'' \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & -4 & 4 & 8 \\ 0 & 0 & -2 & 2 & 4 \\ 0 & 0 & -1 & 9 & 26 \end{array}\right) \xrightarrow{r_3''' \leftrightarrow r_5'''} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & -1 & 9 & 26 \\ 0 & 0 & -2 & 2 & 4 \\ 0 & 0 & -4 & 4 & 8 \end{array}\right) r_3''' = (-1)\cdot r_3'''$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & 1 & -9 & -26 \\ 0 & 0 & -2 & 2 & 4 \\ 0 & 0 & -4 & 4 & 8 \end{array}\right) \begin{array}{l} r_4^{iv} = r_4''' + 2r_3 \\ r_5^{iv} = r_5''' + 4r_3 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & 1 & -9 & -26 \\ 0 & 0 & 0 & -16 & -48 \\ 0 & 0 & 0 & -32 & -96 \end{array}\right) \begin{array}{l} \left(\cdot -\frac{1}{16}\right) \\ \left(-\frac{1}{32}\right) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & 1 & -9 & -26 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \underset{r_5 = r_5 - r_4}{\sim} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 & 6 \\ 0 & 0 & 1 & -9 & -26 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} r(A) = 4 \\ r(A|b) = \min\{4,5\} = 4 \end{array}\right\} r(A) = r(A|b) = 4 \quad \text{μοναδική λύση}$$

Παράδειγμα

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \end{array}\right.$$

Παράδειγμα

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 1 \\ 0x_1 - 2x_2 + 3x_3 + 3x_4 &= 5 \\ 2x_1 - x_2 + 4x_3 + x_4 &= 10 \\ x_1 + 0x_2 + 0x_3 - x_4 &= -1 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 3 & 5 \\ 2 & -1 & 4 & 1 & 10 \\ 1 & 0 & 0 & -1 & -1 \end{array}\right) \begin{array}{l} r_3' = r_3 - 2r_1 \\ r_4' = r_4 - r_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & 3 & 5 \\ 0 & -3 & 2 & 3 & 8 \\ 0 & -1 & -1 & 0 & -2 \end{array}\right) r_2' \leftrightarrow r_4' \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & -1 & -1 & 0 & -2 \\ 0 & -3 & 2 & 3 & 8 \\ 0 & -2 & 3 & 3 & 5 \end{array}\right) (-1)\cdot r_2'$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & -3 & 2 & 3 & 8 \\ 0 & -2 & 3 & 3 & 5 \end{array}\right) \begin{array}{l} r_3'' = r_3' + 3r_2' \\ r_4'' = r_4' + 2r_2' \end{array} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 5 & 3 & 14 \\ 0 & 0 & 5 & 3 & 9 \end{array}\right) r_4''' = r_4'' - r_3''$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 5 & 3 & 14 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right)$$

$0x_1 + 0x_2 + 0x_3 + 0x_4 = -5$

$0 = -5$ αδύνατη

$$\left.\begin{array}{l} r(A) = 3 \\ r(A|b) = 4 \end{array}\right\} \text{αδύνατη}$$

Ομογενή γραμμικά συστήματα

$A\underline{x} = \underline{0} \qquad A \in \mathbb{R}^{m\times n}$

$r(A|0) = r(A)$

Οπότε αν $r(A) = n$ τότε μία μοναδική

Οπότε αν $r(A)=n$ τότε μία μοναδική

$r(A)<n$ έχω άπειρες λύσεις

Παράδειγμα

$$\begin{aligned} 2x_1 + x_2 + 5x_3 &= 0 \\ x_1 - 3x_2 + 6x_3 &= 0 \\ 3x_1 + 5x_2 + 4x_3 &= 0 \\ 0x_1 + 7x_2 - 7x_3 &= 0 \end{aligned} \qquad \begin{pmatrix} 2 & 1 & 5 \\ 1 & -3 & 6 \\ 3 & 5 & 4 \\ 0 & 7 & -7 \end{pmatrix} r_2 \leftrightarrow r_1$$

$$\begin{pmatrix} 1 & -3 & 6 \\ 2 & 1 & 5 \\ 3 & 5 & 4 \\ 0 & 7 & -7 \end{pmatrix} \begin{matrix} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 3r_1 \end{matrix} \begin{pmatrix} 1 & -3 & 6 \\ 0 & 7 & -7 \\ 0 & 14 & -14 \\ 0 & 7 & -7 \end{pmatrix} \begin{matrix} r_3'' = r_3' - 2r_2' \\ r_4'' = r_4' - r_2' \end{matrix} \begin{pmatrix} 1 & -3 & 6 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2 γραμ. 3 στ. άρα έχει άπειρες λύσεις.

$$\begin{aligned} x_1 - 3x_2 + 6x_3 &= 0 \\ 7x_2 - 7x_3 &= 0 \end{aligned} \qquad x_3 = k \in \mathbb{R} \quad , x_2 = k.$$

$x_1 - 3k + 6k = 0 \quad x_1 = -3k.$

$$\begin{pmatrix} -3k \\ k \\ k \end{pmatrix} = k \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \quad , k \in \mathbb{R}$$

Παράδειγμα

Παράδειγμα

$$4x_1+12x_2-7x_3+6x_4=0$$
$$x_1+3x_2-2x_3+x_4=0$$
$$3x_1+9x_2-2x_3+11x_4=0$$

$$\begin{pmatrix} 4 & 12 & -7 & 6 \\ 1 & 3 & -2 & 1 \\ 3 & 9 & -2 & 11 \end{pmatrix} r_1 \leftrightarrow r_2$$

$$\begin{pmatrix} 1 & 3 & -2 & 1 \\ 4 & 12 & -7 & 6 \\ 3 & 9 & -2 & 11 \end{pmatrix} \begin{matrix} r_2' = r_2 - 4r_1 \\ r_3' = r_3 - 3r_1 \end{matrix} \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 8 \end{pmatrix} r_3'' = r_3' - 4r_2'$$

$$\begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

2 γραμ. (2 εξισώσεις) 4 στήλες (4 άγνωστοι)

$2<4$ άπειρες λύσεις.

$$x_1+3x_2-2x_3+x_4=0$$
$$x_3+2x_4=0 \qquad x_4=k\in\mathbb{R}. \quad x_3=-2k.$$

Αν θέσω $x_2=l$, $l\in\mathbb{R}$

$$x_1+3l-2(-2k)+k=0$$
$$x_1+3l+5k=0 \Rightarrow$$
$$x_1=-3l-5k$$

$$\begin{pmatrix} -3l-5k \\ l \\ -2k \\ k \end{pmatrix} \quad k, l \in \mathbb{R}$$

$$\begin{pmatrix} -3l \\ l \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -5k \\ 0 \\ -2k \\ k \end{pmatrix} = l\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + k\begin{pmatrix} -5 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

βάση του διαν. χώρου των λύσεων

Επίλυση πολλών συστημάτων γραμμικών εξισώσεων με κοινό πίνακα συντελεστών.

$A \in \mathbb{R}^{m \times n}$ και $b_1, b_2, \dots b_k \in \mathbb{R}^m$

$$A\underline{x_1} = \underline{b_1}$$
$$A\underline{x_2} = \underline{b_2}$$
$$\vdots$$
$$A\underline{x_k} = \underline{b_k}$$

$$(A \mid b_1 \mid b_2 \dots \mid b_k)$$

Παράδειγμα

$$x_1 + 2x_2 - x_3 = 2$$
$$5x_1 + 6x_2 + 0x_3 = 11$$
$$3x_1 + 0x_2 - 5x_3 = -2$$

$$x_1 + 2x_2 - x_3 = 2$$
$$5x_1 + 6x_2 + 0x_3 = 5$$
$$3x_1 + 0x_2 - 5x_3 = 8$$

$$\left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 2 & 2 \\ 5 & 6 & 0 & 11 & 5 \\ \dots \end{array}\right) \begin{array}{l} r_2' = r_2 - 5r_1 \\ r_3' = r_3 - 3r_1 \end{array} \left(\begin{array}{ccc|c|c} 1 & 2 & -1 & 2 & 2 \\ 0 & -4 & 5 & 1 & -5 \\ 0 & -6 & -2 & -8 & 2 \end{array}\right) \cdot \frac{1}{4}$$

$$\left(\begin{array}{ccc|c|c} 5 & 6 & 0 & 11 & 5 \\ 3 & 0 & -5 & -2 & 8 \end{array}\right) \begin{array}{l} r_2' = r_2 - 5r_1 \\ r_3' = r_3 - 3r_1 \end{array} \left(\begin{array}{ccc|cc} 0 & & & & \\ 0 & -6 & -2 & -8 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|cc} 1 & 2 & -1 & 2 & 2 \\ 0 & 1 & -\frac{5}{4} & -\frac{1}{4} & \frac{5}{4} \\ 0 & -6 & -2 & -8 & 2 \end{array}\right) r_3'' = r_3' + 6r_2' \left(\begin{array}{ccc|cc} 1 & 2 & -1 & 2 & 2 \\ 0 & 1 & -\frac{5}{4} & -\frac{1}{4} & \frac{5}{4} \\ 0 & 0 & -\frac{38}{4} & -\frac{38}{4} & \frac{38}{4} \end{array}\right) -\frac{4}{38}$$

$$\left(\begin{array}{ccc|cc} 1 & 2 & -1 & 2 & 2 \\ 0 & 1 & -5/4 & -1/4 & 5/4 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right) -4 \left(\begin{array}{ccc|cc} 1 & 2 & -1 & 2 & 2 \\ 0 & -4 & 5 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right)$$

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ -4x_2 + 5x_3 &= 1 \\ x_3 &= 1 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ -4x_2 + 5x_3 &= -5 \\ x_3 &= -1 \end{aligned}$$

- Υπολογισμός αντίστροφου πίνακα εναλλακτικά

$$A X = \underline{I} \qquad x = A^{-1} \qquad A[x_1 \mid x_2 \mid x_3 \mid \dots \mid x_n] = [e_1 \mid e_2 \mid \dots e_n]$$

$$\begin{aligned} A \underline{x_1} &= \underline{e_1} \\ A x_2 &= \underline{e_2} \\ &\vdots \\ A x_n &= \underline{e_n} \end{aligned}$$

$$x = [\underline{x_1} \mid \underline{x_2} \mid \underline{x_3} \dots \underline{x_n}]$$

$$I = [\underline{e_1} \mid \underline{e_2} \mid e_3 \mid \dots e_n]$$

$$A^{-1}$$

$$[A \mid \underline{I}] \longrightarrow [\underline{I} \mid x]$$

$$[A \,\vdots\, I] \longrightarrow [I \,\vdots\, X]$$

Παράδειγμα.

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix} \qquad I = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[A \,\vdots\, I] = \left(\begin{array}{ccc:ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \; r_3' = r_3 - 3r_1$$

$$\left(\begin{array}{ccc:ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -6 & 4 & -3 & 0 & 1 \end{array}\right) \; r_3'' = r_3' + 3r_2'$$

$$\left(\begin{array}{ccc:ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 7 & -3 & 3 & 1 \end{array}\right) \; r_3'' \tfrac{1}{7} \quad \left(\begin{array}{ccc:ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{array}\right)$$

$$r_2''' = r_2'' - r_3'' \qquad r_1''' = r_1'' + r_3''$$

$$\left(\begin{array}{ccc:ccc} 1 & 2 & 0 & 4/7 & 3/7 & 1/7 \\ 0 & 2 & 0 & 3/7 & 4/7 & -1/7 \\ 0 & 0 & 1 & -\frac{3}{7} & \frac{3}{7} & \frac{1}{7} \end{array}\right) \; 1/2 \quad \left(\begin{array}{ccc:ccc} 1 & 2 & 0 & 4/7 & 3/7 & 1/7 \\ 0 & 1 & 0 & 3/14 & 4/14 & -1/14 \\ 0 & 0 & 1 & -3/7 & 3/7 & 1/7 \end{array}\right)$$

$$r_1'''' = r_1''' - 2r_2'''$$

$$\left(1 = \frac{7}{7},\ 1 - \frac{3}{7} = \frac{7}{7} - \frac{3}{7} = \frac{4}{7}\right)$$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 2/14 & -2/14 & 4/14 \\ 0 & 1 & 0 & 3/14 & 4/14 & -1/14 \end{array}\right.$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/14 & -2/14 & 4/14 \\ 0 & 1 & 0 & 3/14 & 4/14 & -1/14 \\ 0 & 0 & 1 & -3/7 & 3/7 & 1/7 \end{array}\right)$$

$= 7)$

o $A^{-1} = \begin{pmatrix} \frac{2}{14} & -\frac{2}{14} & \frac{4}{14} \\ \frac{3}{14} & \frac{4}{14} & -\frac{1}{14} \\ -\frac{6}{14} & \frac{6}{14} & \frac{2}{14} \end{pmatrix}$

- Μέθοδος Cramer

$A$ αντιστρέψιμος $|A| = \det(A) \neq 0$

$A \underline{x} = \underline{b}$ $\quad A \in \mathbb{R}^{n \times n}$

$x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$, ... $x_n = \frac{D_n}{D}$

$x_1 - 3x_2 + x_3 = 3$

$5x_1 + x_2 - 2x_3 = 1$

$-x_1 + x_2 + 7x_3 = 13$

$A = \begin{pmatrix} 1 & -3 & 1 \\ 5 & 1 & -2 \\ -1 & 1 & 7 \end{pmatrix}$ $\quad \underline{b} = \begin{pmatrix} 3 \\ 1 \\ 13 \end{pmatrix}$

$D = \det(A) = |A| = \begin{vmatrix} 1 & -3 & 1 \\ 5 & 1 & -2 \\ -1 & 1 & 7 \end{vmatrix} =$

$^{1+1} \begin{vmatrix} 1 & -2 \end{vmatrix}$ $\quad ^{1+2} \begin{vmatrix} 5 & -2 \end{vmatrix}$ $\quad ^{1+3} \begin{vmatrix} 5 & 1 \end{vmatrix}$

$$1\cdot(-1)^{1+1}\begin{vmatrix}1&-2\\1&7\end{vmatrix}+(-3)\cdot(-1)^{1+2}\begin{vmatrix}5&-2\\-1&7\end{vmatrix}+1\cdot(-1)^{1+3}\begin{vmatrix}5&1\\-1&1\end{vmatrix}=$$

$$=(1\cdot7-(-2)\cdot1)+3(5\cdot7-(-1)\cdot(-2))+(5\cdot1-(-1)\cdot1)=$$

$$=(7+2)+3\cdot(35-2)+(5+1)=9+99+6=114\neq0$$

$$D_1=\begin{vmatrix}3&-3&1\\1&1&-2\\13&1&7\end{vmatrix}=3\cdot(-1)^{1+1}\begin{vmatrix}1&-2\\1&7\end{vmatrix}+(-3)\cdot(-1)^{1+2}\begin{vmatrix}1&-2\\13&7\end{vmatrix}+$$

$$+1\cdot(-1)^{1+3}\begin{vmatrix}1&1\\13&1\end{vmatrix}=$$

$$3\underbrace{(1\cdot7-(-2)\cdot1)}_{\overset{7+2}{27}}+3\underbrace{(1\cdot7-(-2)\cdot13)}_{\overset{7+26}{99}}+\underbrace{(1\cdot1-13\cdot1)}_{-12}=114$$

$$x_1=\frac{D_1}{D}=\frac{114}{114}=1$$

$$D_2=\begin{vmatrix}1&3&1\\5&1&-2\\-1&13&7\end{vmatrix}=1\cdot(-1)^{1+1}\begin{vmatrix}1&-2\\13&7\end{vmatrix}+3\cdot(-1)^{1+2}\begin{vmatrix}5&-2\\-1&7\end{vmatrix}$$

$$+1\cdot(-1)^{1+3}\begin{vmatrix}5&1\\-1&13\end{vmatrix}=$$

$$=\overset{7+26}{(1\cdot7-(-2)\cdot13)}-3\overset{35-2}{(5\cdot7-(-1)\cdot(-2))}+\overset{65+1}{(5\cdot13-(-1)\cdot1)}=$$

$$=33-99+66=0$$

$= 33 \quad - 99 \quad / \quad 66 \quad = 0$

$$x_2 = \frac{D_2}{D} = \frac{0}{114} = 0$$

$$D_3 = \begin{vmatrix} 1 & -3 & 3 \\ 5 & 1 & 1 \\ -1 & 1 & 13 \end{vmatrix} = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 1 & 13 \end{vmatrix} + (-3) \cdot (-1)^{1+2} \begin{vmatrix} 5 & 1 \\ -1 & 13 \end{vmatrix}$$

$$+ 3 \cdot (-1)^{1+3} \begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} =$$

$$\underbrace{\overset{13-1}{(1 \cdot 13 - 1 \cdot 1)}}_{12} + 3\underbrace{\overset{65 + 1}{(5 \cdot 13 - (-1) \cdot 1)}}_{198} + 3\underbrace{\overset{5+1}{(5 \cdot 1 - (-1) \cdot 1)}}_{18} =$$

$= 228$

$$x_3 = \frac{228}{114} = 2$$

$$\underline{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$